ALGEBRAIC GEOMETRY: HOMEWORK 8

This homework is due on Friday, October 11 by 5pm.

(1) Let $C \subset \mathbb{P}^2$ be an irreducible curve of degree 4 with singularities at $[1 : 0 : 0]$, $[0 : 1 : 0]$, and $[0 : 0 : 1]$. Prove that C is rational.
Hint: Use the Cremona transformation.

(2) Let X and Y be two irreducible varieties that are birationally isomorphic. Prove that there exist non-empty open subsets $U \subset X$ and $V \subset Y$ such that U and V are isomorphic.

(3) Write down an isomorphism of fields

$$\mathbb{C}(s, t) \to \operatorname{frac}\left(\mathbb{C}[x, y, z]/(x^3 + y^3 + z^3 + 1)\right).$$

You should define the map by writing where each generator goes, and describe how you obtained the map. But you need not write down the inverse. (I tried, and the formulas are horrendous. If you have the appetite and some facility with computer algebra, try it!)

(4) (Food for thought. Not to be turned in.) The isomorphism you wrote above most probably involved some roots of unity. That raises the question: are the fields $\operatorname{frac} \mathbb{Q}[x, y, z]/(x^3 + y^3 + z^3 + 1)$ and $\mathbb{Q}(s, t)$ isomorphic?