ALGEBRAIC GEOMETRY: HOMEWORK 7

This homework is due on Friday, October 4 by 5pm.

(1) Show that the map $X \mapsto \overline{X}$ gives a bijection between closed subsets of $\mathbb{A}^n$ and closed subsets of $\mathbb{P}^n$ that do not contain any irreducible component in the hyperplane $V(X_n)$. Here, as usual, $\mathbb{A}^n \subset \mathbb{P}^n$ is the open subset where $X_n \neq 0$.

(2) Prove that every rational map $\mathbb{P}^1 \dashrightarrow \mathbb{P}^n$ extends to a regular map $\mathbb{P}^1 \to \mathbb{P}^n$.

Remark. In particular, any birational automorphism of $\mathbb{P}^1$ is an actual ("biregular") automorphism. It is easy to show that any automorphism of $\mathbb{P}^1$ is a projective linear transformation.

(3) Consider the rational map $\chi\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ given by

$$\chi\colon [X : Y : Z] \mapsto [YZ : XZ : XY].$$

(a) Prove that $\chi \circ \chi = \mathrm{id}$.

(b) Find the maximal open subset of $\mathbb{P}^2$ on which χ is regular.

Remark. For every rational map $f\colon X \dashrightarrow Y$, there always exists a maximal (by inclusion) open set in X on which f is regular. Indeed, this open set is simply the union of all U such that (U, f_U) lies in the equivalence class of maps corresponding to f.

(c) Make precise the statement: χ transforms most lines into conics, some lines stay lines, and a few lines are contracted to points.
The map χ is called a "Cremona transformation." The birational automorphism group of $\mathbb{P}^2$ is generated by the projective linear transformations and the Cremona transformation, but this is a hard result.