ALGEBRAIC GEOMETRY: HOMEWORK 3

This homework is due by 5pm on August 16. The parenthetical remarks are not a part of the assignment!

(1) (Finite k-algebras) Let A be a reduced finitely generated k-algebra with finitely many maximal ideals. Prove that A is isomorphic to $k^{\oplus n}$ for some n.

(In contrast, there are a *ton* of finitely generated non-reduced k-algebras even with just one maximal ideal, for example, $A = k[x, y]/(x^a, y^b)$ for various a, b.)

(2) (Connectedness and idempotents) Let X be an affine algebraic set and let $f \in k[X]$ an idempotent (that is, $f^2 = f$).

(a) Assume that f is non-trivial (that is, $f \neq 0, 1$). Show that $V(f) \subset X$ is a non-empty proper subset of X that is both open and closed. *Hint: Consider* $1 - f$.

(b) Conversely, suppose X has a non-empty proper subset that is both open and closed. Produce a non-trivial idempotent in $k[X]$.

(As a result, connected affine algebraic sets correspond to k-algebras without non-trivial idempotents).

(3) (The nodal cubic) Assume $\operatorname{char} k \neq 2$. Consider $X = V(y^2 - x^3 - x^2) \subset \mathbb{A}^2$ and the map $\phi\colon \mathbb{A}^1 \to \mathbb{A}^2$ given by

$$\phi\colon t \mapsto (t^2 - 1, t^3 - t).$$

(a) Show that ϕ maps $\mathbb{A}^1$ to X, is surjective, and is injective except that the pair of points $\{1, -1\}$ map to the same point (0,0).

(b) Show that the map $\phi^*\colon k[X] \to k[\mathbb{A}^1]$ is injective and its image is the subring $\{f \mid f(1) = f(-1)\}$.

(Thus, X is obtained from $\mathbb{A}^1$ by "gluing the two points 1 and -1". Algebraically, this tranlates into the fact that functions on X are functions on $\mathbb{A}^1$ that take the same value at 1 and -1.)

(4) (Affine conics) Assume $\operatorname{char} k \neq 2$. Let $f \in k[x, y]$ be an irreducible polynomial of degree 2. Show that $V(f)$ is isomorphic to either $\mathbb{A}^1$ or $\mathbb{A}^1 \setminus \{0\}$.
Hint: Show that f can be brought into the form $xy - 1$ or $y^2 - x$ by a linear change of coordinates on $\mathbb{A}^2$.