# ALGEBRAIC GEOMETRY: WORKSHOP 4

Today we will classify all quadric hypersurfaces! Let $k$ be an algebraically closed field of characteristic different from 2.

## 1. Projective linear transformations

Let $A$ be an invertible $(n+1) \times (n+1)$ matrix with entries in $k$. Left multiplication by $A$ yields a map

$$A \colon \mathbb{P}^n \to \mathbb{P}^n.$$

You can check that this map is regular and has a regular inverse (given by $A^{-1}$). Such transformations of $\mathbb{P}^n$ are called *projective linear transformations.*

## 2. Classifying quadric hypersurfaces

**Theorem 2.1.** *Every quadric hypersurface in $\mathbb{P}^n$ can be taken to a hypersurface of the form*

$$V(X_0^2 + \cdots + X_i^2)$$

*by a projective linear transformation.*

In particular, there are only finitely many isomorphism classes of quadric hypersurfaces. In contrast, understanding cubic hypersurfaces up to isomorphism has occupied algebraic geometers for centuries, (and we still know very little!)

Let us prove the theorem. Let $F(X_0, \ldots, X_n)$ be a homogeneous polynomial of degree 2.

(1) Show that $F(X)$ can be expressed as

$$F(X) = X^T M X,$$

where $M$ is a symmetric $(n+1) \times (n+1)$ matrix.

(2) How does $M$ change when you make the transformation $X \mapsto AX$?

(3) Recall the right result from linear algebra and finish the classification.

## 3. Degenerate and non-degenerate quadrics

A quadric hypersurface in $\mathbb{P}^n$ is *non-degenerate* if it is projectively equivalent to

$$V(X_0^2 + \cdots + X_n^2),$$

and *degenerate* otherwise. Rephrase this dichotomy in terms of the matrix $M$ above.

Finally, what do degenerate quadric hypersurfaces in $\mathbb{P}^2$ look like?