Regular maps

Recall the notion of regular maps for quasi-affine varieties —

$$X \subset \mathbb{A}^n, \quad Y \subset \mathbb{A}^m$$

open subsets of Zariski closed sets.

Then a map $f : X \to Y$ is regular if for every $a \in X$ there exist $f_1, \ldots, f_m$, $g_1, \ldots, g_m \in k[x_1, \ldots, x_n]$, $g_i(x) \neq 0$ such that

$$f = \left(\frac{f_1}{g_1}, \ldots, \frac{f_m}{g_m} \right)$$

in a neighborhood of a.

Using charts, we extend the definition to arbitrary algebraic varieties

$f : X \to Y$ is regular if it is continuous and for every $a \in X$ there exist (eqv. for every) charts (U, V, φ) on X with $a \in U$ & (U', V', φ') on Y with $f(a) \in V'$ such that the map $\bar{f}$ below is regular.

$$\begin{tikzcd}
U \cap \varphi^{-1}(U') \arrow[r, "f"'] \arrow[d] & U' \arrow[d] \\
\text{Open in } V \arrow[r, "\bar{f}"] & V'
\end{tikzcd}$$

When X and Y are quasi-projective, there is a more user-friendly criterion.

Say $X \subset \mathbb{P}^n$, $Y \subset \mathbb{P}^m$.

<u>Prop</u> $f: X \to Y$ is regular if and only if for every $a \in X$ there exist homog. poly $F_0, \dots, F_m \in k[X_0, \dots, X_n]$ such that not all F_i are 0 at a and

$f = [F_0 : \dots : F_m]$

in a neighborhood of a.

<u>Pf</u>. ($\Rightarrow$) Suppose f is regular.

Let $a = [a_0 : \dots : a_n]$. wlog $a_n \neq 0$.
Then a lies in the affine chart

$$\{X_n \neq 0\} \xrightarrow{\sim} \mathbb{A}^n \text{ of } \mathbb{P}^n.$$

Let $b = f(a) = [b_0 : \dots : b_m]$ wlog $b_m \neq 0$.
Then b lies in the affine chart

$$\{Y_m \neq 0\} \xrightarrow{\sim} \mathbb{A}^m \text{ of } \mathbb{P}^m.$$

Restricting the charts to X & Y gives us charts

$$\begin{tikzcd}
a \arrow[d, mapsto] & \in X \cap \{X_n \neq 0\} \arrow[rr, "f"] \arrow[d, "\wr"] & & Y \cap \{Y_m \neq 0\} \arrow[d, "\wr"] \\
\bar{a} & \mathbb{A}^n \supset U \arrow[rr, "\bar{f}"] & & V \subset \mathbb{A}^m
\end{tikzcd}$$

Since $\bar{f}$ is regular, there exist

$$f_0, \dots, f_{m-1}, \ g_0, \dots, g_{m-1} \in k[x_0, \dots, x_{n-1}]$$

such that $g_i(\bar{a}) \neq 0$ for any i and

$$\bar{f} = \left(\frac{f_0}{g_0}, \dots, \frac{f_{m-1}}{g_{m-1}}\right) \quad \text{around } \bar{a}.$$

Let us convert this back to homog. coordinates.
Set $f_m = g_0 \cdots g_{m-1}$ and rename

$$f_i \leftarrow \frac{f_i}{g_i} \cdot g.$$

Then f is given around a by

$$[x_0 : \dots : x_{m-1} : 1] \mapsto [f_0(x_0, \dots, x_{m-1}) : \dots : f_m(x_0, \dots, x_{m-1})]$$

We are almost done. We just have to homogenise. Set $d = \max \deg f_i$ and

$$F_i = X_m^d f_i\left(\frac{X_0}{X_m}, \dots, \frac{X_{m-1}}{X_m}\right)$$

Then f is given around a by

$$[X_0 : \dots : X_m] \mapsto [F_0(X_0, \dots, X_m) : \dots : F_m(X_0, \dots, X_m)]$$

($\Leftarrow$) is even easier. Suppose we know that f has the stated form around a.
Let $a = [a_0 : \cdots : a_n]$ with $a_n \neq 0$ &
$f(a) = [b_0 : \cdots : b_m]$ with $b_m \neq 0$

Consider the restriction
$$X \cap \{X_n \neq 0\} \cap \{b_m \neq 0\} \xrightarrow{f} Y \cap \{Y_m \neq 0\}$$

The std chart identifies LHS as a quasi-affine in $\mathbb{A}^n$ & RHS as a quasi affine in $\mathbb{A}^m$.
In terms of the charts, the map f looks like

$$\bar{f} : (x_0, \ldots, x_{m-1}) \mapsto \left(\frac{F_0(x_0, \ldots, x_{m-1}, 1)}{F_m(x_0, \ldots, x_{m-1}, 1)}, \ldots, \frac{F_{m-1}(x_0, \ldots, x_{m-1}, 1)}{F_m(x_0, \ldots, x_{m-1}, 1)} \right)$$

which is regular.

$\square$.

Examples:

① $$f : \mathbb{P}^1 \to \mathbb{P}^2$$
$$f : [X:Y] \mapsto [X^2 : XY : Y^2]$$

$$\text{Image} \subset \{ [U:V:W] \mid UW - V^2 \}.$$

Inverse

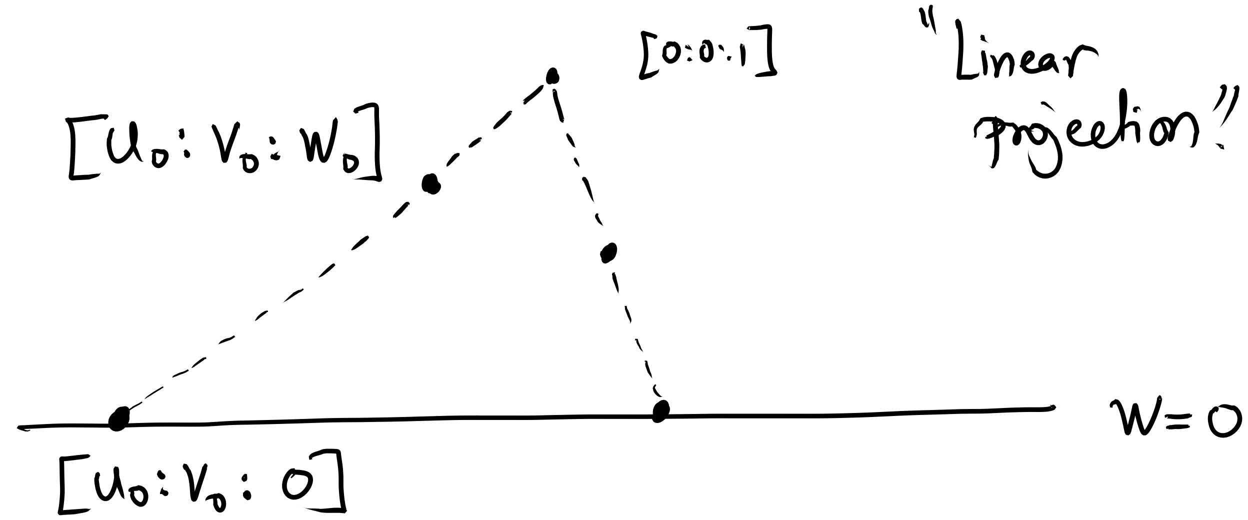
$$g : V(UW - V^2) \longrightarrow \mathbb{P}^1$$

$$\left.\begin{aligned} g : [U:V:W] &\longmapsto [U:V] \\ [0:0:1] &\longmapsto [1:0]. \end{aligned}\right\} \text{ regular!}$$

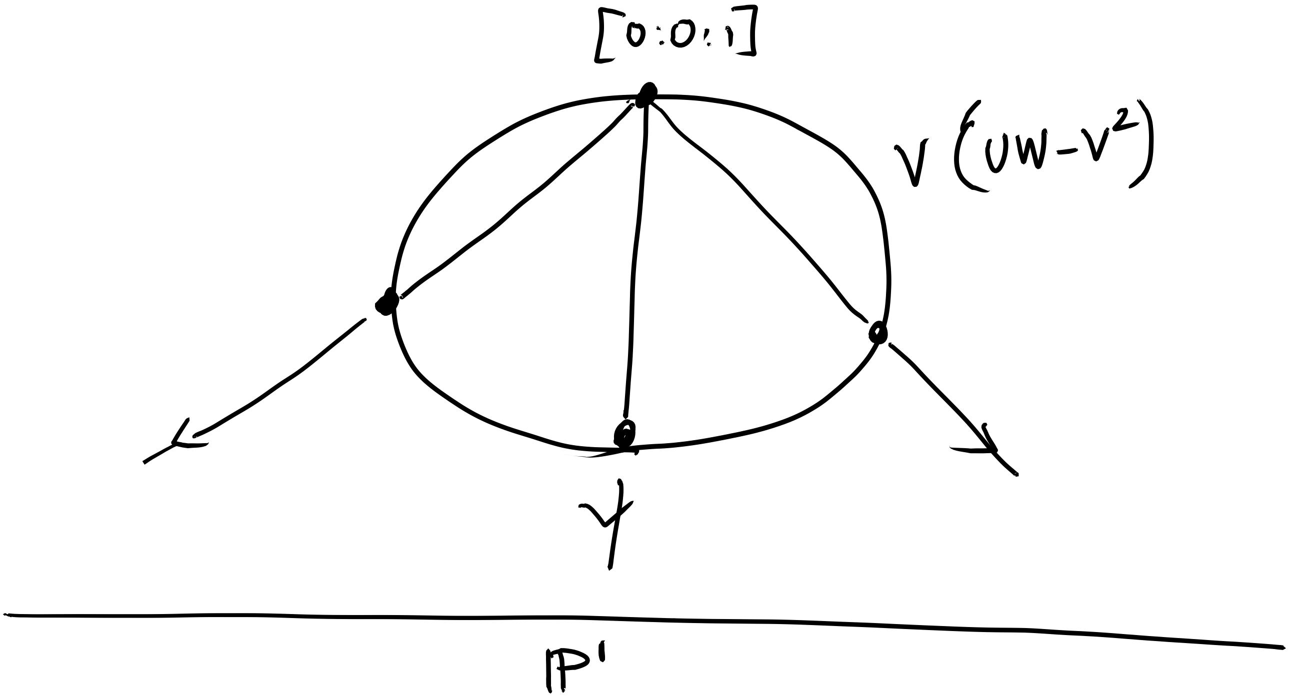
$$\begin{aligned} g &= [U:V:W] \mapsto [U:V] \text{ on } \{W \neq 0\} \\ &= [U:V:W] \mapsto [V:W] \text{ on } \{U \neq 0\}. \end{aligned}$$

Geometry :- What is the map

$$[U:V:W] \longmapsto [U:V] \quad ?$$
$$\mathbb{P}^2 \setminus \{[0:0:1]\} \longrightarrow \mathbb{P}^1$$



"Linear projection"



② $f: \mathbb{P}^1 \to \mathbb{P}^3$

$f: [X:Y] \mapsto [X^3 : X^2Y : XY^2 : Y^3]$ regular!

Image C $\{ [U_0:U_1:U_2:U_3] \mid$

$U_1^2 - U_0U_2,\ U_2^2 - U_1U_3,$
$U_1U_2 - U_0U_3 \} = X$

$g: X \to \mathbb{P}^1$

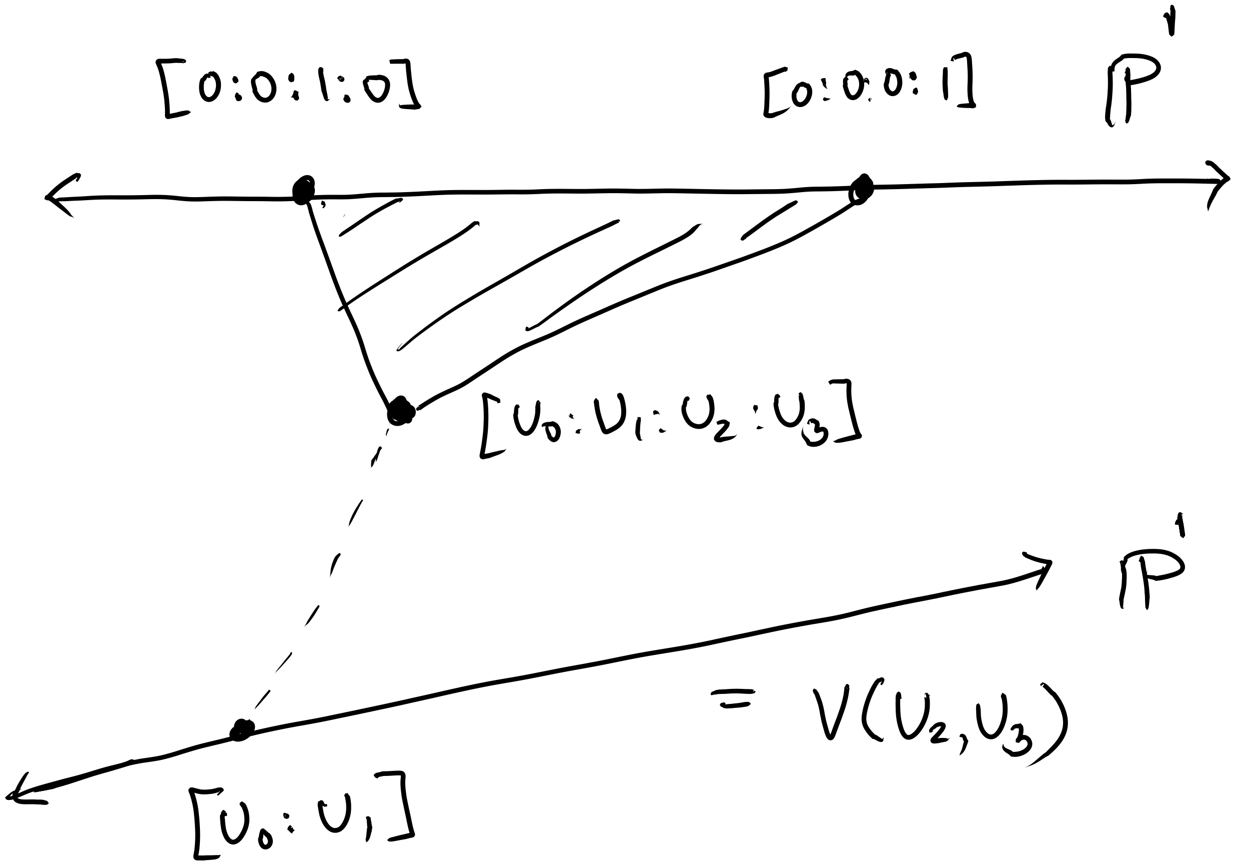
$g: [U_0:U_1:U_2:U_3] \mapsto [U_0:U_1]$ or
$[U_2:U_3]$

is an inverse!

<u>Picture</u> of g:

First $g : [U_0 : U_1 : U_2 : U_3] \mapsto [U_0 : U_1]$

$$\mathbb{P}^3 \setminus \underbrace{V(U_0, U_1)}_{\text{Copy of } \mathbb{P}^1.} \longrightarrow \mathbb{P}^1$$



so g = linear projection with "center of projection" = $V(U_0, U_1)$.

g is not defined along the center of proj. but $g|_X$ extends to $X \cap$ center of proj to a regular map!.

Generalisation

$$f : \mathbb{P}^1 \to \mathbb{P}^n$$
$$[X:Y] \longmapsto [X^n : X^{n-1}Y : \cdots : Y^n]$$

is regular and maps $\mathbb{P}^1$ isomorphically onto
$$\{ [U_0 : \cdots : U_n] \mid U_i U_j - U_l U_k = 0 \text{ if } i+j = l+k \}$$

Def: The image of f is called the rational normal curve in $\mathbb{P}^n$.

No reason to stop at curves

$$v : \mathbb{P}^2 \to \mathbb{P}^5$$
$$v : [X:Y:Z] \mapsto [X^2 : Y^2 : Z^2 : XY : YZ : XZ]$$

Then v is regular.

To find the image, it helps to label the homogeneous coordinates of $\mathbb{P}^5$ by $\{ (i,j,k) \,,\ i+j+k = 2 \,;\ i,j,k \geq 0 \}$

so

$$\mathbb{P}^5 = \{ [u_{(2,0,0)} : u_{(0,2,0)} : u_{(0,0,2)} : u_{(1,1,0)} : u_{(0,1,1)} : u_{(1,0,1)}] \}$$

Then the image lies in

$$X = V(U_I U_J = U_K U_L \mid I+J = K+L).$$

<u>Thm</u>: $V : \mathbb{P}^2 \to X$ is an isomorphism

<u>Pf</u> (Sketch)

- V is a bijection
- X is covered by the charts $\{u_{(2,0,0)} \neq 0\}$, $\{U_{(0,2,0)} \neq 0\}$, $\{U_{(0,0,2)} \neq 0\}$.
- Inverse is given by $[U_I] \mapsto [u_{(2,0,0)} : u_{(1,1,0)} : u_{(1,0,1)}]$ on first chart & likewise on the other two charts.

$\square$

Def: $X \subset \mathbb{P}^5$ is called the Veronese surface.

$v: \mathbb{P}^2 \to \mathbb{P}^5$ is called the (2^{nd}) <u>veronese</u> <u>embedding</u>.

<u>Why</u> stop at 2^{nd} & <u>why</u> <u>stop at a</u> <u>surface</u>?

Define $v_d : \mathbb{P}^n \longrightarrow \mathbb{P}^N$ by

$$[X_i] \mapsto [X^I \mid I = (i_0, \ldots, i_n) \quad i_j \geq 0 \quad \Sigma i_j = d]$$

$$N = \binom{n+d}{n} - 1$$

set $X = V\{[U_I] \mid U_I U_J = U_K U_L$ when $I+J = K+L\}$
$\subset \mathbb{P}^N$.

Thm: $V_d : \mathbb{P}^n \to X$ is an iso.

Pf: Similar to that of $\mathbb{P}^2$ (skipped). $\square$

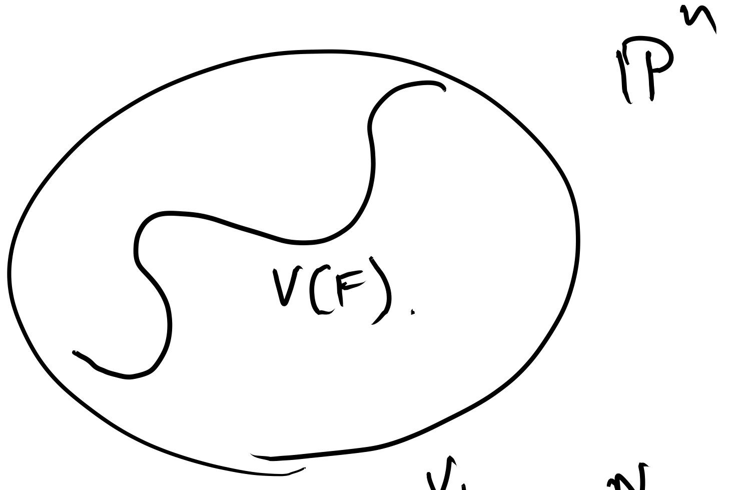
V_d is called the d^{th} Veronese embedding of $\mathbb{P}^n$.

The existence of the Veronese embedding has the following consiquence. Let

$$F = \sum a_I X^I$$

be a homog. poly of dg d in $k[X_0, \dots, X_n]$.

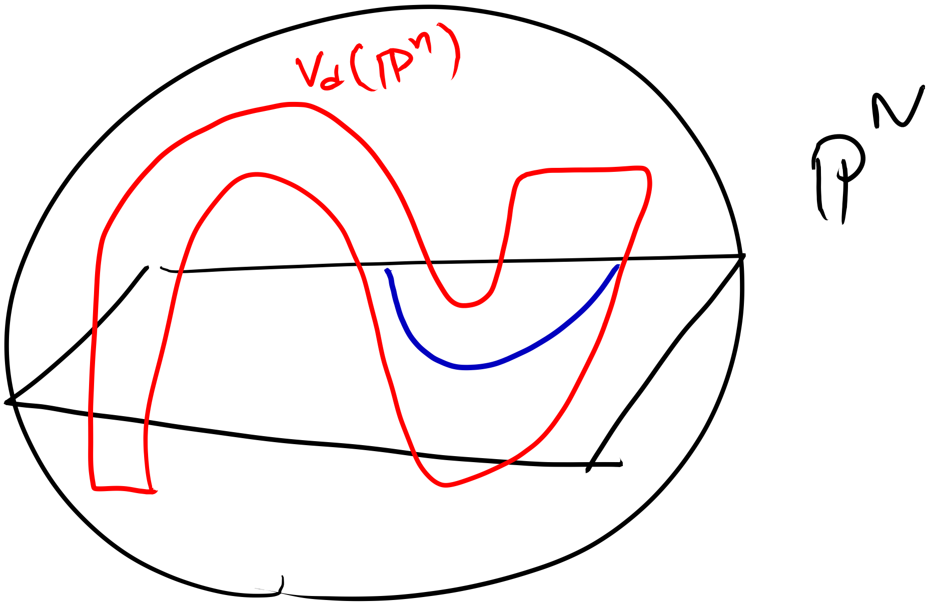
Consider $V(F) \subset \mathbb{P}^n$



Now consider $\mathbb{P}^n \xhookrightarrow{V_d} \mathbb{P}^N$ by the d^{th} veronese. & consider the hyperplane

$$H = V\left(\sum a_I U_I\right) \subset \mathbb{P}^N.$$

Note: $V_d(V(F)) = V_d(\mathbb{P}^n) \cap H.$



Coro: $\mathbb{P}^n \setminus V(F)$ is affine

Pf: By the veronese embedding

$$V_d : \mathbb{P}^n \setminus V(F) \xrightarrow{\sim} \underbrace{(\mathbb{P}^N \setminus H)}_{\mathbb{A}^N} \cap \underbrace{V_d(\mathbb{P}^n)}_{\text{closed}}$$

isomorphism to a closed subspace of affine space!

$\square$

Linear maps, projections, linear sub-spaces.

Suppose $M: k^{n+1} \to k^{m+1}$ is an injective linear map. Then we get a regular induced map

$$M : \mathbb{P}^n \to \mathbb{P}^m.$$

The image of M is a linear subspace of $\mathbb{P}^m$, namely a set cut out by linear (homogeneous) equations.

In fact, the operation of taking the cone gives a bijection

Linear subspaces of $\mathbb{P}^n$ $\longleftrightarrow$ (Nonzero) vector subspaces of k^{n+1}

The <u>smallest</u> linear subspace containing a set $X \subset \mathbb{P}^n$ is called the <u>linear</u> <u>span</u> of X. If $\tilde{x} \in k^{n+1}$ is any non-zero point on the line represented by $x \in X$, then the linear span of X corresponds (under the bijection above) to the vector space span of $\{\tilde{x} \mid x \in X\} \subset k^{n+1}$.

Now suppose $M: k^{n+1} \to k^{m+1}$ has a nonzero kernel $K \subset k^{n+1}$. Then the map

$$M: [x] \mapsto [Mx]$$

is regular on $\mathbb{P}^n \setminus \mathbb{P}K$.

If M is surjective, then M is called the <u>linear</u> <u>projection</u> of $\mathbb{P}^n$ onto $\mathbb{P}^m$ with center $\mathbb{P}K$.