What is algebraic geometry? $\boxed{\text{Lec 1}}$

Algebraic geometry is the study of geometric objects defined algebraically.

Example: Let k be a field, and consider a set of polynomial equations in n variables $x_1, \ldots, x_n$ with coefficients in k, say

$$P_1(x_1, \ldots, x_n) = 0$$
$$\vdots$$
$$P_m(x_1, \ldots, x_n) = 0$$

This gives rise to a geometric object, namely the solution set

$$V = \left\{ (x_1, \ldots, x_n) \in k^n \;\middle|\; \begin{array}{l} P_1(x_1, \ldots, x_n) = 0 \\ \vdots \\ P_m(x_1, \ldots, x_n) = 0 \end{array} \right\}$$

Understanding the geometry of such V is a building block of algebraic geometry.

What does "Understanding the geometry" mean?

① Local geometry – Take $k = \mathbb{C}$, $n = 2$ and $m = 1$ (2 variables, 1 equation)

$$y^2 = x$$

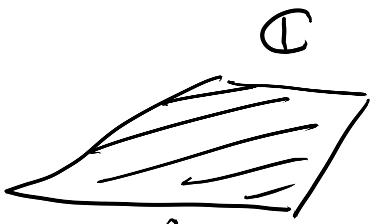
Here the solution set V is in bijection with $\mathbb{C}$ via the map

$$(x,y) \longrightarrow y.$$

The inverse is

$$y \longmapsto (y^2, y)$$

Note that both maps are continuous, so V is homeomorphic to $\mathbb{C}$

$V \cong$ 

Smooth manifold

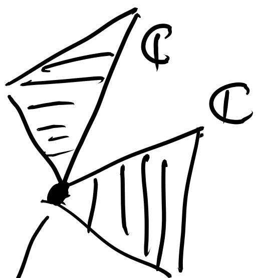
$$y^2 = x^2$$

Here the solution set has two "components"

$V_1 : \; y = x$ or

$V_2 : \; y = -x$

Each one is homeomorphic to $\mathbb{C}$, and they intersect in one point so

$V \cong$ 

Not a smooth manifold

geometry

Q: When is the solution set a smooth manifold?

(2) Global geometry :– Again $k = \mathbb{C}$, and suppose V is a manifold. But which one? What is its Euler characteristic? Cohomology? - - - - -

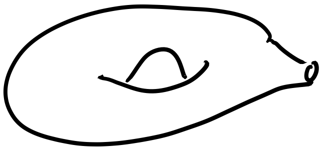
Ex: • $\{y^2 = x\} \cong \mathbb{C} \cong \mathbb{CP}^1 \setminus \{pt\}$.

Do this if time permits • $\{y^2 = x^2+1\} \cong \mathbb{C} \setminus \{0\}$

: $\{(y-x)(y+x) = 1\} \cong \mathbb{CP}^1 \setminus 2$ pts.

• $\{y^2 = x^3+1\} \cong$ Torus $\setminus \{pt\}$

state but don't prove



• $\{y^2 = x^4+1\} \cong$ Torus $\setminus \{2 \text{ pts}\}$

Thm: $V = \{y^2 = f(x)\}$ is smooth iff $f(x)$ has no repeated roots. In this case

$$V \cong \begin{cases} \text{Surf. of genus } \frac{d-1}{2} \setminus 1 \text{ pt} & \text{if } d \text{ odd} \\ \text{Surf of genus } \frac{d}{2} \setminus 2 \text{ pts} & \text{if } d \text{ even} \end{cases}$$

$d = \deg f$.

Why polynomial equations?

(i) Simplest - only use $+, \times, -$

$\Downarrow$

Most widely applicable! Make sense whenever we have a commutative ring.

So we don't have to take $k = \mathbb{C}$.
Can take any ring k.

This has a profound advantages.

$$\boxed{k = \mathbb{C}} \longleftarrow \text{geometric intuition}$$

$$\rightsquigarrow$$

Algebraic formulation

$$\rightsquigarrow$$

Apply geometric ideas to algebraic situations where no obvious geometry exists!

Example: Can define (and use!) genus of $y^2 = f(x)$ where the coeff of f are in $\mathbb{F}_p$!

② A lot of naturally occuring objects <u>are</u> algebraic (defined using polynomial equations).

<u>Ex</u>. Projective spaces
Grassmannians
Flag varieties
Matrix groups
...

Affine algebraic Sets

$\boxed{\text{Lect 2}}$

Let k be a field.
Define $\mathbb{A}^n_k$, called the affine n space over k, by

$$\mathbb{A}^n_k = \{(p_1, \ldots, p_n) \mid x_i \in k\} = k^n.$$

Recall $k[x_1, \ldots, x_n]$ = The ring of polynomials in variables $x_1, \ldots, x_n$ with coeff in k.

Every $f \in k[X_1, \ldots, X_n]$ defines a function

$$\mathbb{A}^n \to k$$
$$P = (p_1, \ldots, p_n) \mapsto f(p_1, \ldots, p_n) = f(p)$$

Let $S \subset k[X_1, \ldots, X_n]$ be any set.
The <u>affine algebraic set defined by S</u> is the set

$$V(S) \subset \mathbb{A}^n$$
$$V(S) = \{ p \in \mathbb{A}^n \mid f(p) = 0 \text{ for all } f \in S\}$$

A subset $Z \subset \mathbb{A}^n$ is called an affine algebraic set if $Z = V(S)$ for some S.

Examples

① $\emptyset$ is algebraic $\quad \emptyset = V(1)$

② $\mathbb{A}^n$ is algebraic $\quad \mathbb{A}^n = V(0)$

③ Single point $\{p\}$

$\{p\} = V(x_1 - p_1, \dots, x_n - p_n)$

④ Algebraic subsets of $\mathbb{A}^1$

$\quad\quad ||$

Finite subsets of $\mathbb{A}^1$ or $\mathbb{A}^1$

Non examples

④ The unit cube in $\mathbb{A}^1_{\mathbb{R}}$

⑤ Rational points in $\mathbb{A}^n_{\mathbb{C}}$.

⑥ ...

Today's main thm: Let $Z \subset \mathbb{A}^n_k$ be an algebraic set. Then there exists a finite set S such that $Z = V(S)$.

This will follow from an algebraic fact – that $k[x_1, \dots, x_n]$ is a Noetherian ring. Let us recall this notion & prove it.

First, recall the def. of an ideal.
Next, recall the def of an ideal generated by a set S.

Proposition: Let $\langle S \rangle$ be the ideal generated by S. Then $V(\langle S \rangle) = V(S)$.

Pf. Show $V(\langle S \rangle) \subset V(S)$ and
$V(S) \subset V(\langle S \rangle)$.
Details skipped. (but will do in class)
$\square$

Recall the def. of a Noetherian ring.
Prop: TFAE
① Every ideal of R is generated by finitely many elements of R. ("finitely generated"
② Every infinite chain of ideals
$$I_1 \subset I_2 \subset I_3 \subset \cdots$$
stabilizes.
Pf: Skip if students seem familiar with this.

Def: A Noetherian ring R is one where the above conditions hold.

Ex. ① $R = \mathbb{Z}$

② $R =$ field.

Thm (Hilbert basis thm): If R is Noetherian, then so is $R[x]$.

Cor: $k[x_1, \dots, x_n]$ is Noetherian.

Pf of thm: Let $I \subset R[x]$ be an ideal. Let us find a finite generating set for I. For every m, define

$$J_m = \{\text{leading coeff}(f) \mid f \in I, \deg(f) \leq m\} \cup \{0\}.$$

Then $J_m \subset R$ is an ideal.
Also $J_m \subset J_{m+1}$. Since R is Noeth. the chain $J_0 \subset J_1 \subset J_2 \subset \cdots$ stabilizes.
Say $J_m = J_{m+1} = \cdots = J$.
Also each $J_1, \dots, J_m$ is finitely gen.
Say $J_l = \langle G_l \rangle$ $\quad G_l \subset R$ finite.
For each $g \in G_l$, let $P_g \in I$ be of deg $\leq l$ such that lead coeff $(P_g) = g$.
We claim that

$$I = \langle P_g \mid g \in G_0 \cup \cdots \cup G_m \rangle.$$

$\supset$ is clear.
For $\subset$, take $f \in I$.
We induct on $\deg(f)$.
If $\deg(f) = 0$, then $f \in J_0 = \langle G_0 \rangle$
and for $g \in G_0$, $P_g = g$, so done.

Otherwise, we use the division alg.
Let $d = \deg(f)$. Say
$$f = a_d x^d + \cdots$$
Then $a_d \in J_d = \langle G_d \rangle$.
Say $G_d = \langle g_1, \ldots, g_r \rangle$ and
$$a_d = c_1 g_1 + \cdots + c_r g_r.$$
Recall we have $P_{g_1}, \ldots, P_{g_r}$ of $\deg \leq d$
such that $g_i = \text{lead coeff}(P_{g_i})$
Then
$$f - c_1 x^{d-\deg(P_{g_1})} P_{g_1} - \cdots - c_r x^{d-\deg(P_{g_r})} P_{g_r}$$
is in I and has <u>lower</u> degree
than f. By induction, it lies in
$$\langle P_g \mid g \in G_0 \cup \cdots \cup G_m \rangle$$
But then f also lies in this set.
The induction step is complete

$\square$.

Conseq: If $Z \subset \mathbb{A}^n$ is algebraic then $Z = V(S)$ for S finite

Pf: $Z = V(T)$ for some T
$= V(\langle T \rangle)$
but $\langle T \rangle = \langle S \rangle$ for some finite S
so
$$Z = V(S).$$
$\square$

If time permits
- Examples of Non-Noetherian rings.
- The ring of continuous functions $\mathbb{R} \to \mathbb{R}$ is not Noetherian.

Prop: R Nothenian, $I \subset R$ ideal
$\Rightarrow R/I$ Noetherian.
i.e. R Noetherian & $R \to S$ surj ring hom. $\Rightarrow S$ Noetherian.
Pf - If there is time.

$\boxed{\text{Caution}}$ - Subrings of Noeth. rings not necessarily Noetherian!

Zariski Topology [Lec 3]

Recall from last time
$\mathbb{A}^n_k = \{ (P_1, \dots, P_n) \mid P_i \in k \}$

$S \subset k[X_1, \dots, X_n] \rightsquigarrow V(S) \subset \mathbb{A}^n$
An affine algebraic set.

These subsets define a topology on $\mathbb{A}^n$ called the Zariski topology. In this lecture we will see how.

Recall the definition of a topology on a set X. Usually it is defined by specifying a collection of subsets of X which are called the "open sets", which satisfy certain axioms. We can equivalently specify a topology by specifying the complements of the open sets, which are called "closed sets".
The collection of closed sets must satisfy the following axioms

1. ϕ and X are closed.
2. Arbitrary intersections of closed sets are closed.
3. Finite unions of closed sets are closed.

Set $X = \mathbb{A}_k^n$ and let the closed sets be the affine algebraic subsets of X, namely the sets $V(S)$ for various S.

<u>Prop</u>: The above defines a topology on X.

<u>Pf</u>: Check all the axioms. The key things are the following –

1. $\phi = V(1) \qquad X = V(0)$.

2. If $X_i = V(S_i)$, then
$$\cap X_i = V(\cup S_i)$$

3. If $X_1 = V(S_1) \quad X_2 = V(S_2)$ then
$$X_1 \cup X_2 = V(S_1 S_2).$$

The first two are easy so I will skip the details in the notes (and decide to do them in class depending on the students).

For ③, remember
$$S_1 S_2 = \{ f_1 f_2 \mid f_1 \in S_1, f_2 \in S_2 \}.$$
easy:
$$X_1 \cup X_2 \subset V(S_1 S_2)$$
For the reverse inclusion, let $p \in V(S_1 S_2)$. We want $p \in X_1 \cup X_2$. Suppose $p \notin X_1$. We want $p \in X_2$.

$p \notin X_1 \Rightarrow \exists\, f_1 \in S_1$ s.t. $f_1(p) \neq 0$.

But $p \in V(S_1 S_2)$

so $f_1 f_2 (p) = 0 \quad \forall\, f_2 \in S_2$

so $f_2(p) = 0 \quad \forall\, f_2 \in S_2$

$\Rightarrow \quad p \in X_2$

$\square$.

③ + induction $\Rightarrow$ finite union property.

So we have a topology.

Ex. $X = \mathbb{A}^1$.

Then the closed sets are $\mathbb{A}^1$ or finite sets.

(so for $k = \mathbb{C}$ this is a much coarser topology than the usual topology).

On the topic of the usual topology.

If $k = \mathbb{R}$ or $\mathbb{C}$, $\mathbb{A}^n = k^n$ already has a topology, namely the usual (Euclidean) topology.

Prop: A Zariski closed/open set is also closed/open in the usual topology.

BUT NOT conversely! The Zariski topology has really few open sets compared to the Euclidean topology.

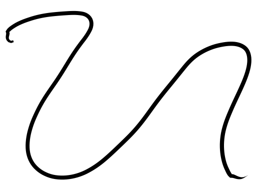
Rem (optional) In general, if k is a topological field, then the same holds. - Zariski topology is always coarser than the topology of $\mathbb{A}^n_k$ coming from k.

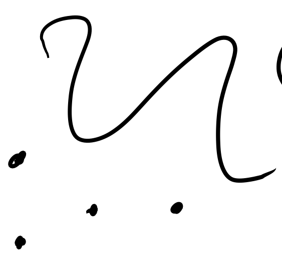
In general, however, the Zariski topology is the only one we have on $\mathbb{A}^n$. So whenever we say closed/open, we now on mean the Zariski top. (otherwise we specify by saying "Euclidean" etc).

<u>Prop</u>: Let $f \in k[x_1, \ldots, x_n]$. Then the function
$$f : \mathbb{A}^n \to \mathbb{A}^1$$
defined by f is continuous.

<u>Pf</u>: Do just for warm up.

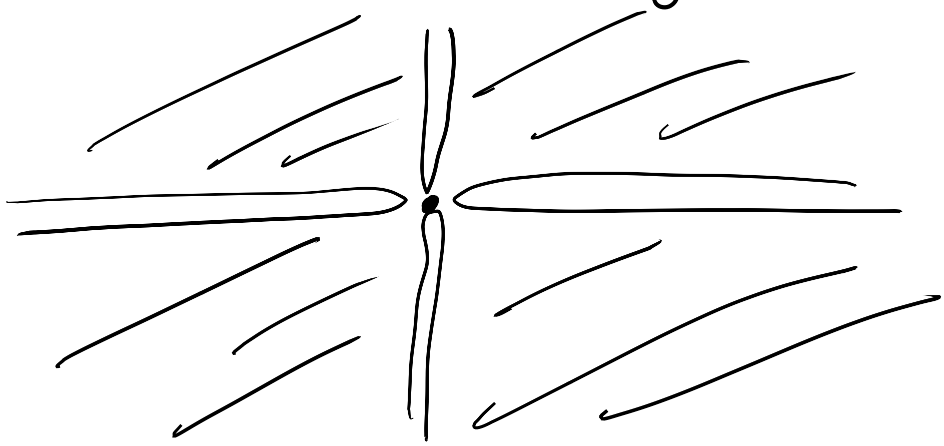
<u>Examples</u>: Zariski closed subsets of $\mathbb{A}^2$.

①  (curve)

②  (curve $\cup$ pts)

<u>Example</u>: A set that is Neither closed <u>nor</u> open

$\mathbb{A}^2$ ∖ axes ∪ origin



(spend some time explaining).

<u>Recall from topology</u>. X a top sp.

Closure of a set S = smallest closed set containing S

S is dense if closure of S is X.

<u>Ex.</u> $S = \mathbb{Z} \subset \mathbb{A}^1_{\mathbb{C}}$

what is the closure of S?

<u>Observe</u>: Zariski topology on $\mathbb{A}^n_k$ is not <u>Hausdorff</u> (if k is infinite).