ALGEBRAIC GEOMETRY: PRACTICE QUESTIONS

You should be able to do all the questions on all the homeworks and all the workshop questions. In addition, here are more practice questions.

(1) Prove that $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^2$ are birational but not isomorphic.

(2) Prove that $\mathbb{P}^1 \times \mathbb{P}^3$ and $\mathbb{P}^2 \times \mathbb{P}^2$ are birational but not isomorphic.

(3) Let $X = V(xy, yz) \subset \mathbb{A}^3$. Decompose X into irreducible components. What are their dimensions?

(4) Let X be an irreducible quasi-projective variety. State the definition of the dimension of X. Show that X has subvarieties of every dimension from 0 to $\dim X$.

(5) Prove that a closed subset $X \subset \mathbb{A}^n$ is irreducible if and only if the ideal $I(X) \subset k[x_1, \dots, x_n]$ is prime.

(6) Show that any birational automorphism of $\mathbb{P}^1$ is a projective linear transformation.

(7) For every $n \geq 2$, give an example of a birational isomorphism $\mathbb{P}^n \dashrightarrow \mathbb{P}^n$ that does not extend to a regular map.

(8) Let M be an $n \times n$ matrix. A vector $v \in k^n$ is called a *generator* for M if the set $v, Mv, M^2v, \dots, M^{n-1}v$ spans k^n. For example, if M is the diagonal matrix with distinct entries $a_1, \dots, a_n$, then the vector $[1, \dots, 1]$ is a generator. Let $U \subset \mathbb{P}\,\mathrm{Mat}_{n\times n}$ be the set (up to scaling) of matrices that admit a generator. Prove that U is Zariski open.
(Hint: Consider the space of pairs (M, v) such that v is not a generator of M.

(9) Let $\phi\colon \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^{(a+1)(b+1)-1}$ be the composite $\tau = \sigma \circ (\nu_a \times \nu_b)$, where the map $\nu_m\colon \mathbb{P}^1 \to \mathbb{P}^m$ is the degree m Veronese and $\sigma\colon \mathbb{P}^m \times \mathbb{P}^n \to \mathbb{P}^{(m+1)(n+1)-1}$ is the Segre map.
   (a) Write the map τ explicitly: where does $([X : Y], [U : V])$ go?
   (b) Let $F \subset k[X, Y, U, V]$ be bihomogeneous of bidegree a, b with $a, b > 0$. Show that $\mathbb{P}^1 \times \mathbb{P}^1 \setminus V(F)$ is affine.

(10) Find all singular points on the curve $V(X^3Y - Z^4) \subset \mathbb{P}^2$.

(11) Let $X \subset \mathbb{P}^n$ be a closed subvariety. Fix positive integers r and l. Let $\Sigma \subset \mathrm{Gr}(r, n+1)$ be the set of Λ such that $\dim(\mathbb{P}\Lambda \cap X) \geq l$. Prove that Σ is a closed subvariety.

(12) Prove that not all degree 4 hypersurfaces in $\mathbb{P}^3$ contain a line.