The Uniform m-lemma

Let F be a coherent sheaf on $\mathbb{P}^n / k$.

Def: We say that F is *m-regular* if $H^i(F(m-i)) = 0$ for all $i > 0$.

Rem: Every sheaf is m-regular for $m \gg 0$. The shift is irrelevant for this remark. But the shift makes inductive arguments smoother.

Prop.: Let F be m-regular. Then

1. F is m'-regular for all $m' \geq m$.
2. $H^0(F(r)) \otimes H^0(\mathcal{O}(1)) \to H^0(F(r+1))$ is surjective. for all $r \geq m$.
3. $F(r)$ is globally generated for all $r \geq m$.

Pf sketch: Induct on n. First, take k to be infinite. Then choose a generic hyperplane $H \subset \mathbb{P}^n$. This gives $0 \to \mathcal{O}(-1) \xrightarrow{h} \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_H \to 0$

Since h is generic, it avoids all associated primes of F. So we get

$$0 \to F(-1) \to F \to F_H \to 0. \quad \text{Twist by } (r-i)$$

On coh. $\cdots \to H^i(F(r-i)) \to H^i(F_H(r-i)) \to H^{i+1}(F(r-i-1)) \to \cdots$

$\Rightarrow$ F_H is r-regular if F is.

$\Rightarrow$ We know ①, ②, ③ for F_H and m. Now chase the cohomology + Serre vanishing $\square$

Thm (Uniform-m lemma): Let P be a polynomial. There exists m depending only on n and P such that every ideal sheaf $\mathcal{I} \subset \mathcal{O}_{\mathbb{P}^n}$ of hilb poly P has regularity m.

Pf. Induct on n. After slicing by a hyperplane, we get

$$0 \to \mathcal{I}(-1) \to \mathcal{I} \to \mathcal{I}_H \to 0.$$

Hilb poly of $I_H =: Q$. Then $Q(l) = P(l) - P(l-1)$, so Q is det. by P.

By ind. hyp. $\exists\, m$ s.t. I_H is m-regular. (m depending only on n, Q).

Twist by $(r-i+1)$ and take cohomology.

We get

$$H^{i-1}(\mathcal{I}_H(r+1-i)) \to H^i(\mathcal{I}(r-i)) \to H^i(\mathcal{I}(r+1-i)) \to H^i(\mathcal{I}_H(r+1-i))$$

Suppose $i \geq 2$, and $r \geq m$. Then both ends vanish by induction.

So $H^i(\mathcal{I}(r-i)) = H^i(\mathcal{I}(r+1-i))$.

$= H^i(\mathcal{I}(r+2-i)) \ldots = 0$ eventually!

$\Rightarrow H^i(\mathcal{I}(r-i)) = 0$. (for $r \geq m$, $i \geq 2$).

<u>Remains</u>: H^1. For H^1, we have

$$H^0(\mathcal{I}(r)) \to H^0(\mathcal{I}_H(r)) \to H^1(\mathcal{I}(r-1)) \to H^1(\mathcal{I}(r)) \to 0$$

$\Rightarrow H^1(\mathcal{I}(x))$ is monotonically decreasing for $x \geq m$.

<u>Claim</u>: If $h^1(\mathcal{I}(r-1)) \neq 0$ then $h^1(\mathcal{I}(r)) < h^1(\mathcal{I}(r-1))$.

<u>Pf</u>: Suppose not. Then $H^0(\mathcal{I}(r)) \to H^0(\mathcal{I}_H(r))$ is surjective.

But then $H^0(\mathcal{I}(r+s)) \to H^0(\mathcal{I}_H(r+s))$ is surjective for all $s \geq 0$!

Then $H^1(\mathcal{I}(r-1)) = H^1(\mathcal{I}(r)) = H^1(\mathcal{I}(r+1)) = \ldots$ ← never vanishes!

$\#$ $\square$.

So $h^1(\mathcal{I}(r))$ ~~monoto~~ strictly decreases for $r \geq m$.

But $h^0(\mathcal{I}(m)) - h^1(\mathcal{I}(m)) = P(m)$ ← <u>Known</u>.

and $h^0(\mathcal{I}(m)) \leq h^0(\mathcal{O}_{\mathbb{P}^n}(m))$.

$\Rightarrow h^1(\mathcal{I}(m)) \leq \underbrace{h^0(\mathcal{O}_{\mathbb{P}^n}(m)) - P(m)}$,

depends only on P and m.

so $h^1(\mathcal{I}(r)) = 0$ for $r \geq m +$

$\square$.

Consequence: We get a pointwise inj. map

$$\underline{\mathrm{Hilb}}^{P}_{\mathbb{P}^n}(k) \hookrightarrow Gr(k).$$

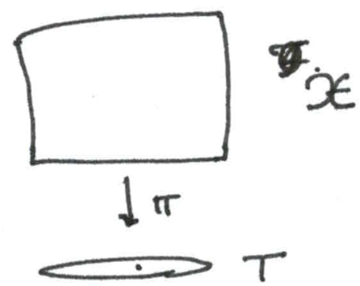
Next: A map of functors.

$$Z \subset \mathbb{P}^n \times T \quad \text{flat over } T.$$

Want: $\pi_* \mathcal{I}_Z(m) \subset \pi_* \mathcal{O}_{\mathbb{P}^n}(m)$ to be a sub-~~vector~~ bundle. with locally free quotient $\pi_* \mathcal{O}_Z(m)$.

Know: Exactly what happens fiberwise

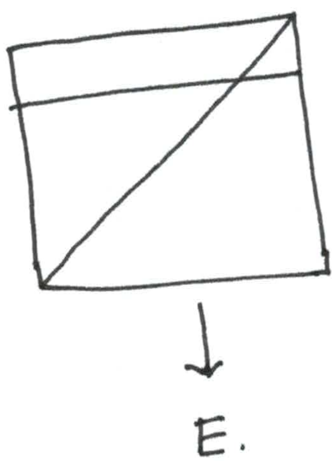
Setup: F a coherent sheaf on $\cancel{\mathbb{P}^n \times T}$ $\mathcal{X}$ flat over T.



$(R^i \pi_* F) \otimes k(t)$ and

$H^i(\mathcal{X}_t, F_t)$.

Example:



$= E \times E \qquad \mathcal{F} = \mathcal{O}(\Delta - p \times E)$.

Then $\mathcal{F}$ is E-flat.

$\pi_* \mathcal{F} = 0$, so $\pi_* \mathcal{F}|_p = 0$

but $H^0(E, \mathcal{F}_p) \neq 0$.

$\big(H^0(E, F_x) = 0 \ \forall\ x \neq p$ and $\mathbb{C}$ for $x = p$.

$H^1(E, F_x) = 0 \ \forall\ x \neq p$ and $\mathbb{C}$ for $x = p. \big)$

What is $R^1 \pi_* \mathcal{F}$?

Cohomology and base change

Setup. $\mathcal{X} \to T$ projective morphism $\mathcal{F}$ a coh. sheaf on $\mathcal{X}$ flat over T.

Thm: There exists a complex of locally free T-modules of finite rank

$$K^\cdot : \quad 0 \to K^0 \to K^1 \to K^2 \to \cdots \to K^n \to 0$$

such that for any $S \to T$, we have

$$R^i \pi_{S*}(\mathcal{X}_S, \mathcal{F}_S) \cong H^i(K \otimes_{\mathcal{O}_T} \mathcal{O}_S).$$

Motivation.

$$\begin{tikzcd}
\mathcal{X}_t, \mathcal{F}_t \arrow[d] & & \mathcal{X} \quad \mathcal{F} \arrow[d, "\pi"] \\
t \arrow[rr, hook] \arrow[d, squiggly] & & T \arrow[d, squiggly] \\
H^i(\mathcal{X}_t, \mathcal{F}_t) & & R^i \pi_* \mathcal{F}.
\end{tikzcd}$$

General: $\quad S \to T \quad \rightsquigarrow \quad (R^i \pi_S)_* \mathcal{F}_S.$

Pf sketch: Let us show the statement without the "locally free of finite rank". Let $T = \operatorname{spec} A$. Cover $\mathcal{X}$ by affines $\mathcal{U}_i$. We have the Čech complex.

$$C^\cdot : \quad 0 \to \mathcal{C}^0(\mathcal{U}, \mathcal{F}) \to C^1(\mathcal{U}, \mathcal{F}) \to \cdots \to C^n(\mathcal{U}, \mathcal{F}) \to 0$$

which has the property that $\forall\ A \to B$,

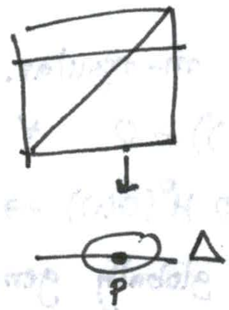
$$R^i \pi_{B*}(\mathcal{F}_B) = H^i(C^\cdot \otimes_A B).$$

Also $C^\cdot$ are flat. The only trouble is that they are not ~~of finite rank~~ finitely generated.

Lemma: Let $C^\cdot$ be a finite complex of flat A-modules whose coh. is finitely generated. Then $\exists$ a finite complex of finitely generated flat A, say $K^\cdot$ ~~modules such that~~ and a map $K^\cdot \to C^\cdot$ which induces iso on cohomology even after any base change $A^- \to B^-$.

Pf: Skip.

Rem: Back to example:



Near p, say Δ, loc. param t.

$$\mathcal{O}_\Delta^m \xrightarrow{M} \mathcal{O}_\Delta^m$$

Generic ~~rank~~ M is an iso $\Rightarrow m = n$

For $t=0$, $\mathrm{rk}\, M$ drops by 1. $\Rightarrow$ In some coordinates

$$M = \begin{pmatrix} t^k & & \\ & 1 & \\ & & \ddots \\ & & & 1 \end{pmatrix}. \Rightarrow \begin{array}{l} R^0\pi_*\mathcal{F} = 0 \\ R^1\pi_*\mathcal{F} = \cancel{\mathcal{O}_\Delta/t^k}\ k[t]/t^i \end{array}$$

[What is i?].

Cor: Semicontinuity.

Cor: If all $h^i(\mathcal{F}) = 0$ for $i > 0$, then $\pi_*\mathcal{F}$ is locally free. and $R^i\pi_*(\mathcal{F}) = 0 \ \forall\ i > 0$.

Flattening Stratification, and the end of the pf of existence of Hilb.

Where we are: Given $\mathbb{P}^n$, and polynomial P.

For $m \gg 0$, we get a natural transformation $\underline{\mathrm{Hilb}}^P \to Gr(r_m, V_m)$,

$$\begin{array}{c} Z \subset \mathbb{P}^n_S \\ \downarrow \\ S \end{array} \quad \rightsquigarrow \quad \pi_* \mathcal{I}_Z(m) \subset \pi_* \mathcal{O}_{\mathbb{P}^n_S}(m) = V_m \otimes \mathcal{O}_S .$$

We know that the transformation is injective on k-points. Today we'll prove that it is represented by a locally closed subscheme of Gr.

On Gr: $\quad 0 \to S \to V_m \otimes \mathcal{O}_{Gr} \to Q \to 0$.

On $Gr \times \mathbb{P}^n$:

$$\begin{tikzcd} \pi_1^* S \arrow[r] \arrow[dr] & V_m \otimes \mathcal{O}_{Gr\times\mathbb{P}^n} \arrow[d] \\ & \mathcal{O}_{Gr\times\mathbb{P}^n}(m). \end{tikzcd} \qquad \begin{tikzcd} Z \arrow[r, phantom, "\subset"] \arrow[dr] & \mathcal{O}_{Gr\times\mathbb{P}^n} \arrow[d] \\ & Gr. \end{tikzcd}$$

i.e. $\quad \pi_1^* S(-m) \to \mathcal{O}_{Gr\times\mathbb{P}^n}$.

Let $\mathcal{I}$ = image of this map, and Z the corresponding subscheme.

Fiberwise Z_t is defined by the r_m-hom. poly spanning S_t for $t \in Gr$.

$\underline{\mathrm{Hilb}}^P$ ~~maps to the locus~~ should be the locus where Z/Gr is flat with hilb poly P.

Thm: Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^n \times S$. There exist finitely many polynomials $\{P\}$ and locally closed subschemes $\{S^P\}$ of S such that

(1) $S = \coprod S^P$

UP: (2) $\mathcal{F}$ is flat over S^P with Hilb poly P.

(3) if $T \to S$ is a map such that $\mathcal{F}_T$ is flat over T with hilb poly P then $T \to S_P$.

Rem: Taking $S = Gr$, $\mathcal{F} = \mathcal{O}_Z$, we get a transformation (by (2)) $S_P \to \mathrm{Hilb}_P$. We also get a transformation $\mathrm{Hilb}_P \to S_P$ (by (3)).

It is easy to check that these are mutually inverse.

Sketch of Pf :

Case : $n=0$. i.e. $\mathcal{F}$ is a coherent sheaf on S. Then flat $\Leftrightarrow$ locally free.

Preliminary reductions : Suffices to prove the statement locally on S.
(Gluing is automatic by UP.)

Let $s \in S$ be a closed point. $\exists$ open $U \subset S$ containing s such that we have a presentation

$$\mathcal{O}_U^m \xrightarrow{M\cdot} \mathcal{O}_U^n \to \mathcal{F} \to 0,$$

where $n = \dim(\mathcal{F}|_s)$.

Claim : For $T \to U$, the pull-back $\mathcal{F}_T$ is free of rank n iff all entries of M are zero on T.

So the stratum of the flattening stratification on U containing s is given by the vanishing of M.

General Case :

Fact : $\mathcal{F}$ on $\mathbb{P}^n_{\times}S$ is flat iff $\pi_*(\mathcal{F}(m))$ are locally free for $m \gg 0$.

Idea : Use the above with the $n=0$ case. Let $\mathcal{F}$ be coh. on $\mathbb{P}^n_{\times}S$.

Claim : $\exists$ $N \gg 0$ s.t. $\forall$ $r \geq N$ we have

(1) $R^i\pi_*\mathcal{F}(r) = 0$ $\forall$ $i>0$

(2) $H^i(\mathcal{F}_t(r)) = 0$ $\forall$ $i>0$, $t \in S$

(3) $R^0\pi_*\mathcal{F}(r)|_t \to H^0(\mathcal{F}_t(r))$ an iso.

NOT trivial.
Uses generic flatness.
Hard algebra.
Will not prove.

Let $\{W_I\}$ be a common flat. strat. of $\pi_*\mathcal{F}(N+0), \dots, \pi_*\mathcal{F}(N+n)$.

$I = (r_0, r_1, \dots, r_n)$. $\leftrightarrow$ Polynomials P of deg n. (interpolation).

$\{W_I\} = \{W_P\}$. Now, for $s \geq n$, consider the flat. strat of $\pi_*\mathcal{F}(N+0), \dots, \pi_*\mathcal{F}(N+s)$. The underlying sets are the same as $\{|W_P|\}$ but there may be additional equations. Restricting to $|W_P|$ for some P, each s gives an ideal sheaf $\mathcal{I}_s$ on W_P with

$$\mathcal{I}_s \subset \mathcal{I}_{s+1} \subset \mathcal{I}_{s+2} \subset \dots \quad \leftarrow \text{must stabilize.}$$

The stabilized ideal sheaves define the correct scheme structure on $|W_P|$, and the resulting locally closed strata are the required ones $\square$.

Pathalogical Examples.

① Hilbert schemes often parametrize objects that we were not so eager to parametrize. In fact, there may be entire components consisting of "degenerate objects" and the dimension of these components often exceeds the dimension of the components that parametrize "nice" objects.

$$\underline{\text{Ex}}: \quad \mathrm{Hilb}^{3mH}_{\mathbb{P}^3} \supset \underbrace{\{\text{Twisted Cubics}\}}_{12 \text{ dim}} \cup \underbrace{\overline{\{\text{Plane Cubics} \amalg \text{Pt}\}}}_{15 \text{ dim}}$$

② The Hilbert schemes are often extremely pathalogical schemes. (i.e. reducible, non-reduced, etc, not of expected dim.).

Furthermore, such pathalogical behavior may occur at [illegible] points that parametrize "nice" subschemes.

Mumford's Example: Nonreduced Component of a Hilb scheme.

Consider $C \subset S$, where $S \subset \mathbb{P}^3$ is a smooth cubic. $/\mathbb{C}$.

and $[C] = [4H+2L]$, where $L \subset S$ is a line

Exercise: Show that (a) $|4H+2L|$ is a base-point free linear system on S and its general member is smooth and irreducible.

(**Hint**: Using $S \cong \mathbb{P}^2$ blown up at 6 points, $H = 3h - E_1 - \cdots - E_6$,

$L = E_1$; so $4H+2L = 12h - 4E_2 - \cdots - 4E_6$) — (May or may not help)!

(In fact $|H+L|$ is base point free).

Then $\deg C = H\cdot(4H+2L) = 14$

$2g-2 = (4H+2L)(3H+2L) = 36-4+14 = 46 \Rightarrow g = 24.$

So Hilb poly of $C = 14m - 24 + 1 = 14m - 23.$

The dimension of

$\mathcal{P} := \{ C \subset \mathbb{P}^3 \mid C$ lies on a smooth cubic surface of class $4H+2L \}$.

$= \dim(\text{space of cubic surfaces}) + \dim |4H+2L|$.

$= \binom{3+3}{3} - 1 + \frac{1}{2}(4H+2L)(5H+2L)$ $\quad [\chi(L) = \frac{1}{2}L(L-K)]$

$= 19 + \frac{1}{2}(60+8+10-4) = \underline{\underline{56}}$.

<u>Claim</u>: $\mathcal{P}$ is irreducible

<u>Pf (Sketch)</u>: $\{ (S,L) \mid S \subset \mathbb{P}^3$ smooth cubic, $L \subset S$ line $\}$ is irreducible.

$\mathcal{P}$ fibers over this set, and the fibers are open subsets of $\mathbb{P}^{37}$. $\Rightarrow \mathcal{P}$ is irreducible.

<u>Note</u>: $\mathcal{P}$ contains all sm. curves of deg 14, genus 24 on sm. cubics.

<u>Q</u>: Is $\overline{\mathcal{P}}$ a component of the Hilb scheme?

Let $C \subset \mathbb{P}^3$ be a smooth curve of deg 14 and genus 24.

$$\begin{array}{ccc} H^0(\mathbb{P}^3, \mathcal{O}(3)) & \longrightarrow & H^0(C, \mathcal{O}(3)) \\ \| & & \| \\ 20 & & 19 + h^0(K_C(-3)) \end{array}$$

$\deg K_C(-3) = 46 - 42 = 4 \Rightarrow$ could be effective. $\Rightarrow$ ☹

Suppose C does not lie on a cubic.

$$\begin{array}{ccc} H^0(\mathbb{P}^3, \mathcal{O}(4)) & \longrightarrow & H^0(C, \mathcal{O}(4)) \\ & & \| \\ 35 & & 33 \end{array}$$

$\Rightarrow C$ lies on a pencil of quartics. $\langle Q_1, Q_2 \rangle$

Q_1, Q_2 are irreducible $\Rightarrow Q_1 \cap Q_2 = C \cup D \quad \deg(D) = 2$.

<u>Ex.</u> $p_a(D) = 0$. Conclude that D is a plane conic (not necessarily reduced or irreducible).

So, we can get $C \subset \mathbb{P}^3$ of deg 14 genus 24, not lying on a ~~Σ~~ cubic by :–

① Choose a plane conic D

② Choose a penicil of quartics containing D $\langle Q_1, Q_2 \rangle$

③ $C := Q_1 \cap Q_2 \setminus D$.

Dim count : $3+5=8$

Pencil of quartics : $Gr(2, H^0(\mathcal{I}_D(4))) = Gr(2, \binom{4+3}{3} - (9))$
$= Gr(2, 26) = \cancel{44}\ 48$

Total : 56.

Conclusion : A general member of $\mathcal{P}$ is not a specialization of curves ~~out~~ on cubics. $\Rightarrow \overline{\mathcal{P}}$ is a component of Hilb.

Now, N_{C} sits in

$$0 \to N_{C/X} \to N_C \to N_{X/\mathbb{P}^3}|_C \to 0 \qquad (4H+2L)^2 = 48+16-4 = 60$$

$$0 \to \mathcal{O}_C(C) \to N_C \to \mathcal{O}(3)|_C \to 0 \qquad H^1(\mathcal{O}_C(C)) = 0.$$

$$\Rightarrow H^0(N_C) = 60-24+1 + 42-24+1 + h^1(\mathcal{O}_C(3))$$
$$= 37 + 19 + h^0(K_C(-3)) \qquad K_C = -H+C$$
$$h^0(\mathcal{O}(2L)) \qquad = 3H+2L$$
$$= 57$$

$\Rightarrow \mathcal{P}$ is ~~generi~~ everywhere non-reduced !.

Murphy's Law : There is no geometric possibility so horrible that it cannot be found on a Hilb scheme.

Holds for : Smooth curves in proj space.
smooth surfaces in $\mathbb{P}^5$.