Local Study of the Hilbert Scheme.

Sept 16, 2014

Ref :- • Kollar, "Rational Curves on Algebraic Varieties" (I.2)
• Background – " BLR – Néron Models"
– " Rational points on algebraic varieties – Poonen "

Goal: Use the Hilbert functor to understand local properties of the Hilbert scheme.

Last time: X a projective scheme over a field k.
$Z \subset X$ a subscheme $\leftrightarrow$ a point $z \in \mathrm{Hilb}(X)$.

Thm: We have an isomorphism of vector spaces

$$T_z \mathrm{Hilb}(X) \cong \mathrm{Hom}_Z(I_Z/I_Z^2, \mathcal{O}_Z) = \mathrm{Hom}_X(I_Z, \mathcal{O}_Z).$$

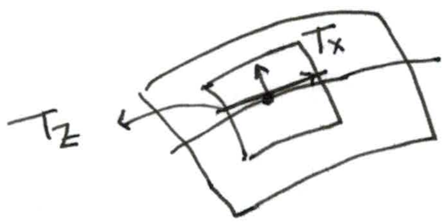
Def: The sheaf $\mathcal{H}om_Z(I_Z/I_Z^2, \mathcal{O}_Z)$ is called the "normal sheaf" of Z in X.

Rem: Suppose both Z and X are smooth. Then we have the sequence

$$0 \to I_Z/I_Z^2 \to \Omega_{X/k}|_Z \to \Omega_{Z/k} \to 0$$

Dualizing

$$0 \to T_{Z/k} \to T_{X/k} \to N_{Z/X} \to 0.$$



$N_{Z/X} = T_X / T_Z$.

Then $T_z \mathrm{Hilb}(X) = H^0(Z, N_{Z/X})$.

$T_z \mathrm{Hilb}(X)$ || "First order def of Z in X."

$H^0(Z, N_{Z/X})$ || "Normal vector field."

Smoothness :

We saw how to compute the dimension of the tangent space to Hilb at a point using the functor. Can we compute the dimension of the Hilbert scheme itself? This turns out to be surprisingly difficult in general. We always have the inequality

$$\dim T_x \mathrm{Hilb} \geq \dim_x \mathrm{Hilb}.$$

and we know that equality holds if Hilb is smooth at x.

It turns out that smoothness can be detected functorially.

Infinitesimal Lifting Criterion

Let X be a scheme locally of finite type over an alg. closed field k.

~~Def. Art~~

Def : A "small extension" of Artinian k-algebras is a surjection $\tilde{A} \to A$ of Art. k algebras with kernel $\cong k$. (as $\tilde{A}$-module).

Alternatively : Kernel $\cong k^m$ (as A-module).

Eg: 1) $k[t]/t^{n+1} \to k[t]/t^n \to 0$.

2) $k[x,y]/(x^2, xy, y^2) \to k[x,y]/(x^2, y)$.

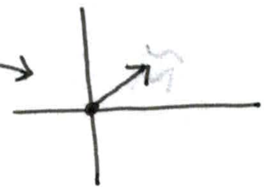
Prop : X/k is smooth if and only if for every small extension of Artinian k-algebras $\tilde{A} \to A$, ~~and~~ any morphism $\operatorname{spec} A \to X$ extends to a morphism $\operatorname{spec}(\tilde{A}) \to X$.

$$\begin{tikzcd} \operatorname{Spec} \tilde{A} \arrow[r, dashed] & X \arrow[d] \\ \operatorname{Spec} A \arrow[u, hook] \arrow[ur] & \operatorname{Spec} k \end{tikzcd}$$

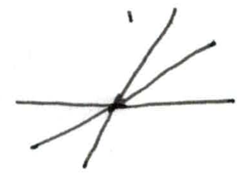
Rem: This is a functorial condition (phrased in terms of $\mathrm{Maps}(-,X)$).

Illustration: Consider $X = \mathrm{spec}\, k[x,y]/xy$.

Not smooth.



Ex: Do a similar analysis for $k[x,y]/(x^3-y^3)$.



$$\begin{tikzcd} \mathrm{Spec}\left(k[\varepsilon]/\varepsilon^2\right) \arrow[r] \arrow[d, hook] & \mathrm{spec}\, k[x,y]/xy \\ \mathrm{Spec}\left(k[\varepsilon]/\varepsilon^3\right) \arrow[ur, dotted, "\times"'] & \end{tikzcd}$$

$X \mapsto \varepsilon$

$Y \mapsto \varepsilon$

$X \mapsto \varepsilon + a\varepsilon^2$

$Y \mapsto \varepsilon + b\varepsilon^2$

$XY \Rightarrow \varepsilon^2 \neq 0.$

Singularity detected!

Remark: If we want to check smoothness at a point $x \in X$, we have to check $\mathrm{spec} A \to X$ whose closed points maps to x.

Pf of thm: Suppose $x = \mathrm{im}(\mathrm{spec} A) \in X$. and X is smooth at x.

Then $\hat{O}_{X,x} = k[[x_1, \dots, x_n]]$

So any map $\hat{O}_{X,x} \to A$ lifts to $\hat{O}_{X,x} \to \tilde{A}$.

Conversely, suppose the lifting criterion holds.

Let $R = k[[m_x/m_x^2]] = k[[x_1, \dots, x_n]]$. we have a map $\hat{O}_{X,x} \to R/m_R^2$ iso on tang. spaces.

By successively lifting maps, we get a $\hat{O}_{X,x} \to R$, which is an iso on tangent spaces.

Then it must be an iso. $\square$.

Remk: The infinitesimal criterion can be used to test (or define!) the notion of a smooth map $f: X \to S$ between schemes.

(Not today).

Application to the Hilbert scheme :

Setup : A an Artin ~~ring~~ local k-algebra with residue field k (not nec. alg cl.).

$\tilde{A} \to A$ a surjection with kernel $\cong k$.

$\mathrm{Ker} = \{ a\varepsilon \mid a \in k \}$.

$$\begin{array}{lccc} \text{Map} & \operatorname{spec} A \to \mathrm{Hilb}_X & \longleftrightarrow & Z_A \subset X_A := X \times \operatorname{spec} A \\ & \cap & & \\ \text{Lift} & \operatorname{spec}\tilde{A} \to \mathrm{Hilb}\,X & \longleftrightarrow & Z_{\tilde{A}} \subset X_{\tilde{A}} := X \times \operatorname{spec}\tilde{A} \end{array}$$

$$\text{s.t.} \quad Z_{\tilde{A}} \times_{\tilde{A}} A = Z_A .$$

Write Z_0 for $Z_A \times_{A} k$, the "central fiber."

Convention : subscript o means restriction to the central fiber.

Equivalently, want to lift $I_{Z_A} \subset \mathcal{O}_{X_A}$ to $I_{Z_{\tilde{A}}} \subset \mathcal{O}_{X_{\tilde{A}}}$ with flat quotient. (over $\tilde{A}$).

Strategy : ① Do the problem of lifting locally. ② Glue. } Both steps present problems.

$R =$ ~~fin. gen. k algebra~~, ~~Ring (Noetherian)~~. Fin gen k-algebra

$I \subset R \otimes_k A$ an ideal. s.t. R_A / I is A-flat

Want to understand ideals $\tilde{I} \subset R_{\tilde{A}}$ lifting I s.t. $R_{\tilde{A}} / \tilde{I}$ is $\tilde{A}$-flat.

Same as before :- classify the "ε- tails" accompanying $f \in I$.

$\tilde{I}$ lifts I. $\Leftrightarrow$ $\forall$ $f \in I$, there exists $g \in R$ such that "$f + \varepsilon g$" $\in \tilde{I}$

⚠ What does $f + \varepsilon g$ mean (as an element of $R \otimes \tilde{A}$) ?
εg is OK. But there is no ring map $R_A \to R_{\tilde{A}}$.
So we cannot treat elts of R_A as elts of $R_{\tilde{A}}$.

Let $i: R_A \to R_{\tilde{A}}$ be a set theoretic lift of $R_{\tilde{A}} \to R_A$.

(e.g. $k[t]/_{t^n} \to k[t]/_{t^{n+1}}$

$a_0 + a_1 t + \cdots + a_{n-1}t^{n-1} \mapsto a_0 + a_1 t + \cdots + a_{n-1}t^{n-1}$).

① $\forall f \in I \ \exists g \in R$ s.t. $i(f) + \varepsilon g \in \tilde{I}$.

Prop: Suppose $\tilde{I} \subset R_{\tilde{A}}$ is an ideal lifting $I \subset R_A$, where R_A/I is A-flat. Then $R_{\tilde{A}}/\tilde{I}$ is $\tilde{A}$-flat iff $\forall f \in I$, there is a unique $g \in R$ (modulo I_0) s.t. $i(f) + \varepsilon g \in \tilde{I}$.

Pf. It suffices to check that

$$0 \to k \xrightarrow{\varepsilon} \tilde{A} \to A \to 0$$

remains exact after $\otimes R_{\tilde{A}}/\tilde{I}$.

Issue: $R_0/I_0 \xrightarrow{\varepsilon} R_{\tilde{A}}/\tilde{I}$ must be inj.

$\Leftrightarrow \forall g \in R_0$ s.t. $\varepsilon g \in \tilde{I}$, we have $g \in I_0$.

So, if $i(f) + \varepsilon g_1$ and $i(f) + \varepsilon g_2 \in \tilde{I}$, then $g_1 \equiv g_2 \mod I_0$

$\square$.

Thus a flat lift gives a function ("tails")

$$\varphi : I \to R_0/I_0 \qquad \tilde{I} = \{ i(f) + \varepsilon g \mid g \equiv \varphi(f) \text{ in } R_0/I_0 \}.$$
$$f \mapsto g.$$

The issue is that this is not R_A-linear, and it is nontrivial to characterize which functions give ideals.

$\tilde{I}$ is closed under $+$ if φ is a group hom. ✓.

Suppose $i(f) + \varepsilon g \in \tilde{I}$ and $(x + \varepsilon y) \in R_{\tilde{A}}$

Then $(i(f) + \varepsilon g)(x + \varepsilon y) \in \tilde{I}$

$\Leftrightarrow \quad i(f)i(x) + \varepsilon(g_0 x_0 + f_0 y) \in \tilde{I}$

$\Leftrightarrow \quad i(f)i(x) + \varepsilon(g_0 x_0) \in \tilde{I}$

$\Leftrightarrow \quad i(fx) + \varepsilon(g_0 x_0 + \text{error}(f,x)) \in \tilde{I}$.

NOT linear

$\Leftrightarrow \varphi(fx) = x\varphi(f) + \text{error}(f,x)$

$\forall f \in I, x \in R_A$.

BUT: If $\tilde{I}_1$ and $\tilde{I}_2$ are two lifts corresponding to
$\varphi_1 : I \to R_0/I_0$ and $\varphi_2 : I \to R_0/I_0$
and $\delta\varphi = \varphi_1 - \varphi_2$, then

$$\delta\varphi(fx) = f\,\delta\varphi(x) \qquad \forall f \in I,\ x \in R_A.$$

i.e. $\delta\varphi \in \mathrm{Hom}_{R_A}(I, R_0/I_0)$.

Conversely if $\tilde{I}_1$ is a lift corresponding to φ_1 and $\delta\varphi \in \mathrm{Hom}_{R_A}(I, R_0/I_0)$, then $\varphi_2 = \varphi_1 + \delta\varphi$ also defines a lift.

Prop: The set of lifts of $I \subset R_A$ to $\tilde{I} \subset R_{\tilde{A}}$ with $\tilde{A}$-flat quotient is either empty or a Principal Homog. space under
$$\mathrm{Hom}_{R_A}(I, R/I_0) = \mathrm{Hom}_{R_0}(I_0, R_0/I_0).$$

Def: We say that $Z_A \subset X_A$ is locally unobstructed if there is an open affine cover of X_0 ~~such~~ over which $Z_A|_U \subseteq X_A|_U$ can be extended to $Z_{\tilde{A}}|_U \subset X_{\tilde{A}}|_U$.