Oct 9 : Moduli of curves

Functor : $M_g : S \mapsto \left\{ \pi : C \to S \quad \begin{matrix} \text{flat proper} \\ \text{geometric fibers are} \\ \text{smooth connected} \\ \text{curves of genus } g \end{matrix} \right\} \Big/ \text{iso.}$

i.e. $\begin{matrix} C_1 & & C_2 \\ \downarrow & \sim & \downarrow \\ S & & S \end{matrix}$ is $\exists$
$$\begin{tikzcd} C_1 \arrow[rr, "\sim"] \arrow[dr] & & C_2 \arrow[dl] \\ & S & \end{tikzcd}.$$

i.e. $M_g(\mathbb{C})$ = Isomorphism classes of complex smooth proj curves.

Prop. : M_g is not representable.

Pf : Let us work over $\mathbb{C}$. Consider a rep. functor $F = \mathrm{Maps}(-, X)$.

Then F forms a sheaf. i.e.

Suppose Y is a scheme & $Y = \bigcup U_i$ open cover

$f, g : Y \to X$ two maps s.t.

$f|_{U_i} = g|_{U_i}$, then $f = g$.

Holds for Zariski open cover. Also holds for analytic, i.e. euclidean open covers. Let us show that this does not hold for M_g.

Idea : Let C be a curve with a non trivial automorphism.

e.g. $C \xrightarrow{2:1} \mathbb{P}^1$ $\quad \sigma : C \to C$ involution

① trivial family $\Leftrightarrow$ $f : Y \to M_g$

Y 

② twisted family $\Leftrightarrow$ $g : Y \to M_g$.

On open covers $f = g$. but on Y, $f \neq g$. !

Actual: $X =$ ~~nodal curve~~

$$Y = \mathbb{C}^* \qquad \tilde{Y} = \mathbb{C}^* \qquad \begin{array}{c}\tilde{Y} \to Y. \\ z \mapsto z^2.\end{array}$$

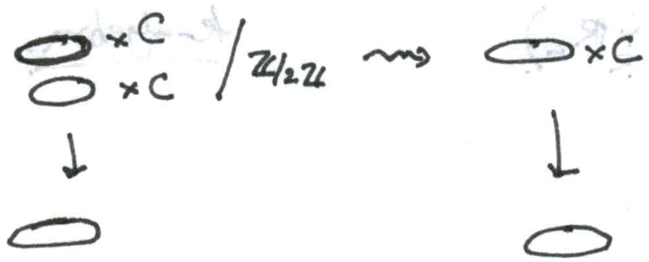
$\tilde{Y} \to Y$ covering space, $\mathbb{Z}/2\mathbb{Z}$ = Deck transf. group.

$$(\tilde{Y} \times C)/\mathbb{Z}_2 \qquad \sigma: (z, x) \mapsto (-z, \sigma(x)).$$

$$\downarrow \quad \leftarrow \text{all fibers isomorphic to } C.$$

$$Y.$$

In fact on a small open disc



Locally a trivial family.

But globally not a trivial family.

Very actual.: $C: \quad y^2 - f(x)$ in affine equation.

Family 1 : $C \times \mathbb{C}^* = \cancel{\mathbb{C}[t, t^{-1}, x, y]/} \; (y^2 - f(x)) \times \mathbb{C}^*$

$$\downarrow$$

$$\mathbb{C}^*$$

Family 2 : $y^2 - t f(x)$. $\leftarrow$ all fibers isomorphic to C.

$$\downarrow$$

$$\mathbb{C}^* = \operatorname{spec} \mathbb{C}[t, t^{-1}].$$

In fact. $y^2 - t^2 f(x)$. $\cong$ trivial family.

$$\begin{array}{c}\mathbb{C}^* \longrightarrow \mathbb{C}^* \\ t \mapsto t^2\end{array}$$

$\Rightarrow$ original family was locally trivial, but globally non-trivial.

General : $\tilde{Y} \to Y$ principal G- bundle and $G \curvearrowright C$.

$(\tilde{Y} \times C)/G \longrightarrow Y$. a locally trivial but globally non trivial family.

Arithmetic:

Consider the $\mathbb{R}$-curves $y^2 = (x^6+1) = C_1 \Leftrightarrow \mathbb{R}$-point of M_g. P_1

$y^2 = -(x^6+1) = C_2 \Leftrightarrow \mathbb{R}$-point of M_g P_2

But over $\mathbb{C}$, $C_1 \cong C_2$.

$\Rightarrow$ $P_1 = P_2$ as $\mathbb{C}$ points of M_g
but $P_1 \neq P_2$ as $\mathbb{R}$-points of M_g } Cannot happen!.

$\square$.

Problem: M_g. not rep.

Enlarge the category of schemes so that M_g becomes representable.

+ make sense of

- sheaves, q coherent sheaves, cohomology
- intersection theory
- sep / prop. etc.

For this bigger category.

"Alg Stacks"

best approximation "Keel-Mori thm".

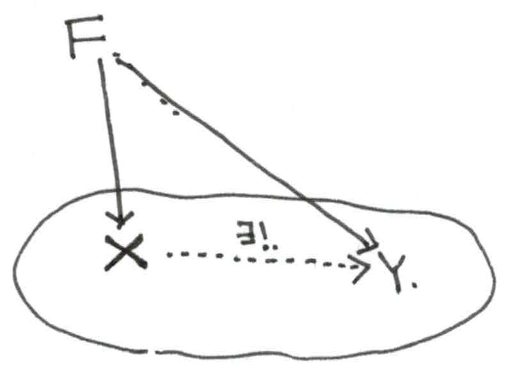
~~Find the "closest"~~
Work with the scheme that ~~is as close / is~~ is the best approximation.
||
"Coarse"-moduli space.

Def: Let $F : Sch^{op} \to Sets$ be a functor. A coarse space for F is a scheme X with a natural transformation $\Psi : F \to Maps(-, X)$ such that

(1) Ψ is a bijection on $\mathbb{C}$-valued points (or k-valued pts for an algebraically closed k).

(2) Given any other Y and $\Psi' : F \to Maps(-, Y)$ $\exists!$ $\varphi : X \to Y$ such that $\Psi' = \varphi \circ \Psi$.

Picture:



Schemes.

X = Closest "shadow" of F in schemes.

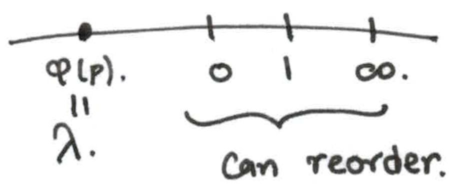
Rem: (2) Characterizes X uniquely up to a unique iso.
(1) is then an additional condition.

Example: M_1 = moduli of genus 1 curves.

C
$\downarrow$
$S = \operatorname{Spec} \mathbb{C}$.

Pick $p \in C$ $\mathcal{L} = \mathcal{O}(2p)$. $H^0(\mathcal{L}) \cong \mathbb{C}^2$

$\varphi : C \longrightarrow \mathbb{P}^1$ branched at 4 pts.



Can reorder.

$\lambda \in \mathbb{P}^1 \setminus \{0, 1, \infty\}$. $\circlearrowleft S_3$

$$\left\{ \lambda, 1-\lambda, \frac{1}{\lambda}, \frac{1}{1-\lambda}, \frac{-\lambda}{1-\lambda}, \frac{\lambda-1}{\lambda} \right\}, \quad j = 256 \frac{(1-\lambda+\lambda^2)^3}{\lambda^2(1-\lambda^2)} \in \mathbb{C}.$$

$j(\lambda) = j(\lambda')$ iff $\lambda \sim \lambda'$ under S_3.

So $M_1(\mathbb{C}) \cong$ Pts of $\mathbb{A}^1_j$

Claim: $\mathbb{A}^1_j$ is the coarse moduli space for M_1.

Pf:

$$\begin{array}{ccc} C & & C \\ \downarrow & & \downarrow \uparrow \sigma \\ S & \xleftarrow{\text{open}} & U \end{array}$$

$\mathcal{L} = \mathcal{O}(2\sigma)$.

$$\begin{tikzcd} C \arrow[r, "2:1"] \arrow[dr] & \mathbb{P}^1_U \arrow[d] \\ & U \end{tikzcd}$$

$C = y^2 - x(x-1)(x-\lambda)$

λ a reg. function on U.

define $j : U \to \mathbb{A}^1$

Then j does not depend on choices

$\Rightarrow$ get $j : S \to \mathbb{A}^1$.

Why is j the initial object?

$\square$

Thm: There exists a quasi-projective coarse moduli space for M_g.

Dim count C, $\mathcal{L}$ = line bundle of deg $d > 2g-2$.

$H^0(L) = d-g+1 = r+1$.

$C \longrightarrow \mathbb{P}^r$ Hilb poly det. by d, g.

$\underline{\text{Hilb}}_{d,g}$ = ~~open~~ Hilbert scheme of genus g deg. d curves in $\mathbb{P}^r$ (Open subset of the full hilb scheme).

$\underline{\text{Hilb}}_{d,g} \longrightarrow M_g$.

$\downarrow$

dim $= h^0(\text{Normal})$.

$$0 \to T_C \to T_{\mathbb{P}^r}|_C \to N \to 0$$

$\text{rk}\, N = r-2$ $\quad$ $\deg N = (r+1)d - (2-2g)$
$= (r+1)d + 2g - 2$

$$\chi(N) = (r+1)d + 2g - 2 + (r-2)(1-g)$$
$$= r(d-g+1) + d + 3g - 3 = (r+1)^2 + 4g - 4$$

Fiber dim $= g + (r+1)^2 - 1$.

$\Rightarrow \dim M_g = 3g-3$

$\square$