Algebraic Stack: Roughly speaking, an algebraic stack is a stack that "locally" looks like a scheme (or equivalently, the spectrum of a ring). I.e., we must have an "atlas" \( U \overset{\pi}{\to} \mathcal{X} \) for \( \mathcal{X} \), where \( \pi \) is a covering in the appropriate sense, and \( U \) is a scheme.

1. \( \pi \) étale \( \iff \) Deligne-Mumford stack
2. \( \pi \) smooth \( \iff \) Artin stack.

However, to make sense of such properties for \( \pi \), it must be representable. Let us see what this entails.

\[
\begin{align*}
U \times_{\mathcal{X}} V &\to U \quad \text{schematic} \quad \forall \xi \in \mathcal{X}(U) \\
\downarrow &\downarrow \chi \\
V &\to \mathcal{X}
\end{align*}
\]

\( U \times_{\mathcal{X}} V \,(T) = \{ (f:T \to U, g:T \to V, \psi : f^*\xi \to g^*\beta) \} \)

\( =: \text{Isom} \,(\xi, \beta) \).

Rem: \( \mathcal{X} \) stack \( \Rightarrow \text{Isom} \,(\xi, \beta) \) is a sheaf (the first condition in the def.)

Prop: We have \( U \times_{\mathcal{X}} V = (U \times V) \times_{\mathcal{X} \times \mathcal{X}} \mathcal{X} \)

\[\text{If:}\]

\[
\begin{array}{ccc}
\square &\to& U \times V \\
\downarrow & & \downarrow \\
\mathcal{X} &\Delta& \mathcal{X} \times \mathcal{X}
\end{array}
\]

\( \square \,(T) = \{ (f:T \to U, g:T \to V, \tau \in \mathcal{X}(T), (f^*\xi, g^*\beta) \to (\tau, \eta)) \} \)

\( \downarrow \text{equiv.} \)

\( \{ (f, g, f^*\xi \to g^*\beta) \} = U \times_{\mathcal{X}} V \,(T) \).

So, \( U \times_{\mathcal{X}} V \) will be a scheme if \( \Delta \) is representable.

In fact the converse is also true.
**Def:** A Deligne-Mumford stack is a stack $\mathcal{X}$ such that

1. $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable, separated, quasi-compact
2. There is a scheme $U$ and an étale surjective morphism $U \to \mathcal{X}$

(called an "atlas").

**Rem:** $DM \leftrightarrow \text{étale atlas}$

$\text{Artin} \leftrightarrow \text{smooth atlas}$

**Rem:** For a DM stack $\mathcal{X}$

**Rem on the diagonal:**

$\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$

either $\phi \leftarrow \text{Isom}(\alpha, \beta) \to \text{Spec } \mathbb{C}$

or $\text{Aut}(\alpha)$

For a scheme, $\Delta$ is a (locally closed) embedding.

**Prop:** Let $\mathcal{X}$ be a DM stack. Then $\Delta$ is unramified.

*Proof:* (In particular, for any $x \in \mathcal{X}(k)$ the group $\text{Aut}(x)$ is finite. In fact, the automorphism scheme is finite and reduced.)

**Pf:**

For $\text{Mg}_g$, we know (1).

We also know that $\Delta$ is unramified.
Prop: Let \( \mathcal{X} \) be a stack over a Noetherian base scheme \( S \), such that

**Thm:**

1. \( \Delta \) is repr., g.c., separated, and unramified.
2. \( E \) is finite type over \( S \) and a smooth surj \( E \to \mathcal{X} \).

Then \( \mathcal{X} \) is D.M.

i.e. Given that \( \Delta \) is unramified, a smooth atlas \( \Rightarrow \) an étale atlas.

Cor: \( Mg \) is a D.M. stack.

**PF:** we only need to produce a smooth atlas.

Let \( d > 2g-2 \). Consider \( H \subset \text{Hilb} \) the open set parameterizing smooth curves of arithmetic genus \( g \) and degree \( d \) in \( \mathbb{P}^r \), where \( r = d-3g+1 \).

Claim: \( H \to Mg \) is smooth.

**PF:** Sufficient to check the infinitesimal lifting criterion.

\[
\begin{array}{c}
E \to E' \\
\downarrow \downarrow \\
\text{Spec} \mathcal{A} \to \text{Spec} \mathcal{A'}
\end{array}
\]

Given: An embedding \( E \subset \mathbb{P}^r_{\mathcal{A}} \) of deg \( d \).

Want: An extension \( E' \subset \mathbb{P}^r_{\mathcal{A}'} \).

Let \( L_{\mathcal{A}} = \mathcal{O}(1) \subset \mathbb{P}^r_{\mathcal{A}} \) restricted to \( E \). We have an iso.

\[ A^{TM} \cong H^0(E, L_{\mathcal{A}}). \]

Extend \( L_{\mathcal{A}} \) to a line bundle \( L_{\mathcal{A}'} \) on \( E' \).

Then \( H^0(E', L_{\mathcal{A}'}) \) is locally free of rank \( (r, \text{th}) \) by coh \( \text{th} \) base change.

so we get \( H^0(E', L_{\mathcal{A}'}) \to A^{TM} \) extending \( \circ \).

Thus \( E' \to \mathbb{P}^r_{\mathcal{A}'} \). (embedding automatic). \( \square \).
Pf 1. thm (Sketch): In char 0, or say over C. 

Take a point \( x: \text{spec} \ C \to \mathcal{X} \). We want to produce an étale chart for \( \mathcal{X} \) around \( x \). We have: 

\[
\begin{array}{ccc}
\mathcal{U}_x & \xrightarrow{i} & \mathcal{U} \\
\downarrow & & \downarrow \\
\mathcal{X} & \Delta & \mathcal{X} \times \mathcal{X}
\end{array}
\]

Note that: 

\[
\begin{array}{ccc}
\mathcal{U}_x & \xrightarrow{i} & \mathcal{X} \times \mathcal{U} \\
\downarrow & & \downarrow \\
\mathcal{X} & \to & \mathcal{X} \times \mathcal{X}
\end{array}
\]

\( \Rightarrow \ i: \mathcal{U}_x \to \mathcal{U} \) is unramified. \( \Rightarrow \)

\[ \tilde{\mathcal{U}}_x \xrightarrow{i} \tilde{\mathcal{U}} \]

étale \( \downarrow \) \( \downarrow \) \( \downarrow \) \( \downarrow \) \( \downarrow \) étale 

\[ \tilde{\mathcal{U}}_x \xrightarrow{i} \mathcal{U} \]

Since \( \mathcal{U} \to \mathcal{X} \) is smooth, \( \tilde{\mathcal{U}}_x \to \text{spec} \ C \) is smooth. 

\( \Rightarrow \) \( \tilde{x} \in \tilde{\mathcal{U}}_x \) is cut out by a regular sequence \( t_1, \ldots, t_m \). 

Lift \( t_i \) to \( \tilde{t}_i \) on \( \tilde{\mathcal{U}} \) and set \( \mathcal{Z} = V(\tilde{t}_i) \subset \tilde{\mathcal{U}} \).

Claim: The map \( \mathcal{Z} \to \mathcal{X} \) is étale over \( \tilde{x} \).

Pf: We need to check that the map \( \sim \mathcal{U}_x \mathcal{Z} \to \mathcal{Z} \) is étale over \( \tilde{x} \).

\[
\begin{array}{ccc}
\sim \mathcal{U}_x \mathcal{Z} & \to & \mathcal{Z} \\
\downarrow & & \downarrow \\
\mathcal{U} & \to & \mathcal{X}
\end{array}
\]

Now \( \tilde{\mathcal{U}}_x \mathcal{Z} \to \tilde{\mathcal{U}} \) is smooth and \( \tilde{\mathcal{U}}_x \mathcal{Z} \to \tilde{\mathcal{U}}_x \tilde{\mathcal{U}} \) is defined by the vanishing of \( \tilde{t}_1, \ldots, \tilde{t}_m \).
Furthermore, over $\tilde{\mathcal{U}} \subseteq \mathcal{U}$ we have

$$
\tilde{\mathcal{U}} \xrightarrow{\alpha} \mathcal{U} \xrightarrow{\mathcal{X}} \tilde{\mathcal{U}} \xrightarrow{\mathcal{X}} \mathcal{U}
$$

def. by

t_1, \ldots, t_n.

$$
\mathcal{U} \xrightarrow{\mathcal{X}} \mathcal{U} \xrightarrow{\mathcal{X}} \mathcal{U}
$$

$\Rightarrow$ by the Jacobian criterion that $\mathcal{U} \xrightarrow{\mathcal{X}} \tilde{\mathcal{U}}$ is smooth of rel. dim $\mathcal{U}$ (i.e. étale) in a neighborhood of $\tilde{\alpha} \subseteq \tilde{\mathcal{U}}$.

Thus $\mathcal{U} \xrightarrow{\mathcal{X}} \mathcal{X}$ is étale over $\alpha$.

Examples:
1. $G$ étale group scheme over $S$

$\Rightarrow BG$ is a DM stack.

2. $G$ smooth group scheme $S$ acting on $X$. such that the stabilizers of geometric points $g \in X$ are geometrically finite and reduced. (automatic in char 0).

Then $[X/G]$ is a DM stack.

$$
\text{Stab}_x \xrightarrow{\Delta} X \xrightarrow{\Delta} [X/G] \xrightarrow{\Delta} [X/G] \xrightarrow{\Delta} [X/G] \xrightarrow{\Delta} [X/G]
$$

$X_{\Delta} \xrightarrow{\Delta} G \xrightarrow{\Delta} X \xrightarrow{\Delta} X \xrightarrow{\Delta} X$