Moduli of curves - Oct 21

Recall our first attempt at formulating $M_g$ as a functor:

$M_g : \text{Schemes} \rightarrow \text{Sets}$

$S \mapsto \left\{ \pi : C \rightarrow S \mid \text{sm prop. curve} \right\} / \text{iso}$

Loss of information comes to bite us while gluing...

$C_i \cap C_j$

$C_i \cup C_j \overset{\text{iso classes}}{\mapsto} \left\{ C_1, C_2 \right\}

\ldots 

\left\{ C_4, C_5 \right\}$

may be a choice involved here.

make choices w.r.t.
1. choices don't agree on triple overlaps
2. too many choices agree on triple overlaps.

Result - Not a sheaf.

Motivation for "stacks" - Generalize the notion of sheaves to accommodate objects as above.

Recall scheme = sheaf + locally Spec$R$
Likewise algebraic = stack + locally Spec$R$
Stack

First condition $\mapsto$ maps can be locally defined and "glued?"
Second condition $\mapsto$ locally, a map from $X$ to our object corresponds to a number of regular functions on $X$ with polynomial conditions.

"algebraic"
Def: A groupoid is a category where every arrow is an isom.

Rem: 1) A groupoid with one object "is" a group.
2) A groupoid with a unique arrow between any pair of objects "is" a set.
3) Eqv. a groupoid with trivial "automorphism groups" is a set.

Recall: A sheaf = contrav. set valued functor + gluing cond.
Stack = "groupoid-valued functor" + gluing cond.

Alternative:
Def: Let $S$ be a category. A category fibered in groupoids over $S$ is a category $F$ with a functor $p: F \to S$ such that

\[ \exists X_2 \to X_1 \]
\[ p \quad p \quad p \]
\[ B_2 \to B_1 \]

\[ X_3 \leftarrow X_2 \to X_1 \]
\[ B_3 \to B_2 \to B_1 \]

leads to slightly messy definitions.
Rem 1  \[ X_2 \xrightarrow{f} X_1 ; F \]

\[ B_2 \xrightarrow{s} B_1 \]

Then \( f \) is an iso. iff \( s \) is an iso.

② Given \( B \in S \), consider

\[ F(B) = \text{category whose objects are } X \in F \text{ with } p(X) = B \text{ and morphisms are } \]

\[ X_1 \xrightarrow{f} X_2 \quad \text{et } p(f) = \text{id}_B. \]

Then \( F(B) \) is a groupoid

③ \[ F(B_2) \xrightarrow{\text{!}} X \]

\[ B_2 \rightarrow B_1 \]

So we guarantee the existence of "pullbacks" which are unique up to a unique iso, without insisting on a particular one.

Examples ① If \( F: S^{\text{op}} \rightarrow \text{sets} \) is a functor, then we can make it a CFG: \( \text{obj} = (s, w) \), \( s \in S \), \( w \in F(s) \)

maps: for every \( f: s \rightarrow t \) put

\[ (s, w) \rightarrow (t, w') \text{ if } w = f^* w'. \]

In particular, every scheme gives a CFG.
2. \( \text{M}_S = \text{A CFG over \textbf{Schemes}}. \)

\[ \text{Obj} = (\Pi : C \to S) \]

\[ \text{maps} = \text{Pull back diagrams} \]

\[
\begin{array}{c}
C_1 \xrightarrow{\text{f}} C_2 \\
\Pi_1 \downarrow \square \downarrow \Pi_2 \\
S_1 \to S_2.
\end{array}
\]

3. \( \text{Vect}_n : \text{A CFG over \textbf{Schemes}}. \)

\[ \text{Obj} = (S, E), \ E \text{ is a vect. bundle of rank } n \text{ over } S \]

\[ \text{maps} = S_1 \xrightarrow{f} S_2 + \text{an iso } E_1 \cong f^*E_2 \]

Similarly \( \text{Coh}, \ \mathbb{Q}_{\text{coh}}, \text{IV particularly } \mathbb{R}_{\text{biv.}} \text{ etc...} \)

4. \( G \text{ a group scheme (over } S). \)

\[ \text{BG} : \text{CFG over \textbf{Schemes}}. \]

\[ \text{Objects} : (\pi : P \to T) \text{ a principal } G \text{-bundle} \]

\[ \text{maps} : \text{Pullback diagrams} \]

\[
\begin{array}{c}
P_1 \xrightarrow{\text{f}} P_2 \\
\downarrow \square \downarrow \\
T_1 \to T_2.
\end{array}
\]

5. \( G \text{ a group acting on } X. \)

\[ [X/G] : \text{CFG over \textbf{Schemes}}. \]

\[ \text{Objects} : (\Pi : P \to T, \ f : P \to X) \]

\[ \text{maps} : \text{Pullbacks} \]

\[
\begin{array}{c}
P_1 \xrightarrow{\text{f}} P_2 \to X \\
\downarrow \downarrow \\
T_1 \to T_2
\end{array}
\]

\( \text{a Principal } G \text{-bundle} \)

\( \text{a } G \text{-equiv. map} \)

\[ \text{Comm. w/ maps to } X. \]
G) $G_0$ : "Universal curve"

Objects - $(T; C \to S, \sigma: S \to C)$

$\sigma$ maps

\[ G_1 \to G_2 \]

$T_1 \to T_2$

Def: $F_1$ and $F_2$ two CFG's fibered over $S$.

A map of CFG's, is a functor $f: F_1 \to F_2$ s.t.

\[ F_1 \xrightarrow{f} F_2 \]

\[ p_1 \downarrow G \downarrow p_2 \]

\[ S = S \]

i.e. $p_1 = p_2 \circ f$.

Ex:

1. $G_0 \to Mg$
2. $[X/G] \to BG$
3. $X \to [X/G]$
4. morphisms of schemes for maps of associated CFG's.