Oct 14: Moduli of Curves

We work over $\mathbb{C}$ (not laziness - all arguments today will be topological.)

$M_g =$ Coarse moduli space of smooth proj. curves of genus $g$.

**Thm:** $M_g$ is irreducible

**Pf:** We will construct an irreducible space that maps surjectively onto $M_g$.

\[ H_{d,g} = \left\{ (C, f) \mid C \text{ is a smooth proj. curve of genus } g, f: C \to \mathbb{P}^1 \text{ a simply branched map of deg } d \right\} / \text{iso} \]

\[ \# \text{ branch points} = 2g + 2d - 2 = b \]

\[ \begin{array}{cccc}
H_{d,g} & (c, f) & H_{d,g} \to M_g \\
\phi & \downarrow & \downarrow \\
\text{Sym}^b(\mathbb{P}^1) \setminus \Delta & \text{br}(f). & \text{compact} \\
\end{array} \]

Branched Covers: $C, D$ Riemann Surfaces $f: C \to D$ finite map. $f^* \Delta \subset D$.

Outside a finite set of points $B \subset D$, $f$ is a covering space.

\[ f^{-1}(B) \subset D \quad \text{a covering space of deg } d \]

Pick a base point $o \in D \setminus B$ and label

\[ f^{-1}(o) = \{ 1, 2, \ldots, d \} \]

Lifting loops give a homomorphism

\[ T_1(D \setminus B, x) \to S_d \]

often called the monodromy representation. Changing the labelling by a permutation $\sigma$ changes $m$ to $\sigma m \sigma^{-1}$. Conversely, any homomorphism $m: T_1(D \setminus B, x) \to S_d$ gives a covering space $f: \Delta \to D \setminus B$.

There is a unique way to complete this covering space to a deg $d$ branched cover $C \to D$, where $C$ is a compact R.S.

Indeed, if $\Delta \subset D \setminus B$ is a punctured disk centered at $b \in B$, then

\[ f^{-1}(\Delta^*) = \Delta_1^* \cup \Delta_2^* \cup \ldots \cup \Delta_k^* \]

(union of punctured disks) with $f: \Delta_i \to \Delta^*$ of the form $z \mapsto z^{n_i}$ where $\sum n_i = d$.
we complete this to
\[ \Delta_1 \cup \Delta_2 \cup \ldots \cup \Delta_k \to \Delta \]

Note: In this picture a loop around \( b \) corresponds to the monodromy permutation
\[
\begin{array}{c}
\tau_1 \text{-cycle} \\
\tau_2 \text{-cycle} \\
\vdots \\
\tau_m \text{-cycle}
\end{array}
\]

monodromy = \((12)(3)(456)\)

Rem: \( C \) is connected iff the image of \( m \) is a transitive subgroup of \( S_d \).

Back to \( H_{d,g} \to \text{Sym}^b(\mathbb{P}^1) \setminus \Delta. =: \text{Sym}^b(\mathbb{P}^1)^* \)

Fibers: \( q_b \cdot q_\cdot \):
Over \( (\mathbb{P}^1, \{P_1, \ldots, P_b\}) \)
\[
\{ q: C \to \mathbb{P}^1 \text{ simply branched over } P_i \} \\
\{ \text{image is transitive} \} \\
\{ \text{a loop around } P_i \to \text{transposition} \}
\]

finite set
\[
\text{indep}_b \to \left\{ (\sigma_1, \ldots, \sigma_b) \mid \sigma_i \in S_d \text{ a simple transposition} \quad \Pi \sigma_i = \text{id} \right\}
\]

Use this to make \( H_{d,g} \to \text{Sym}^b(\mathbb{P}^1) \setminus \Delta \) a covering space.

Thus \( H_{d,g} \) becomes a complex manifold (at least). The map to \( M_g \) is holomorphic. In fact \( H_{d,g} \) is a quasi proj. variety -- finite cover of a variety is a variety (Riemann Existence Thm).

Claim: \( H_{d,g} \) is irreducible.

Pf: Enough to show \( H_{d,g} \) is connected (because it is clearly smooth).

Equivalent to showing that the monodromy of the covering space
\[ H_{d,g} \to \text{Sym}^b(\mathbb{P}^1)^* \]
acts transitively on the fibers.
Fix a base point $p = (p_1, ..., p_b) \in (\text{sym}_b \mathbb{R})^*$. Fix loops as shown:

$$\phi_i(p) = \{ (\sigma_1, ..., \sigma_b) \mid \sigma_i \text{ are transpositions}, \text{gen. trans. subgroup} \}$$

$$\conjugation \ = \ \emptyset \ N.$$ 

Take $\gamma$ to be the loop that switches $i$ and $i + 1$.

as shown:

Trace the loops along to lift $\gamma$ to $H_d,q$.

So, $\gamma$ started at

$$(\sigma_1, ..., \sigma_i, \sigma_i^{-1}, ..., \sigma_b)$$

in the std system of loops

ended at

$$(\sigma_1, ..., \sigma_i, \sigma_i^{-1}, ..., \sigma_b)$$ in the non-standard system of loops

$$(\sigma_1, ..., \sigma_i \sigma_i^{-1} \sigma_i, ..., \sigma_b)$$ in the standard system.

Hence $\gamma : N \to N$ maps

$$(\sigma_1, ..., \sigma_b) \mapsto (\sigma_1, ..., \sigma_i \sigma_i^{-1} \sigma_i, ..., \sigma_b). \quad \text{"Braid move."}$$
Any \((\sigma_1, \ldots, \sigma_d) \in \mathbb{N}\) can be brought into the form
\[(12)(12) \cdots (12) (23)(23) (34)(34) \cdots (d-1,d) (d-1,d).\]
by a sequence of braid moves. i.e. the monodromy \(\text{Hom}_g \to \mathbb{P}^{(3g-3)}\) is transitive.

Now, for \(d \gg 0\), the map \(\text{Hom}_g \to \mathbb{P}^{(3g-3)}\) is surjective.
\[\Rightarrow \mathbb{P}^{(3g-3)}\] is irreducible.

\[\square\]

**dim count:** \(\dim \text{Hom}_g = b = 2g + 2d - 2\)

Fibers \# \(M : \text{Hom}_g \to \mathbb{P}^{(3g-3)}\) \(\leftrightarrow \# f : C \to \mathbb{P}^{(3g-3)}\) for a fixed \(C\).

1. Choice \# a line bundle of degree \(d\) on \(C \to \mathbb{P}^{(3g-3)}\)
2. Choice \# two general sections \# \(L \to C \to \mathbb{P}^{(3g-3)}\)

so that \(f = [s_1 : s_2]\)

(up to scaling)

fiber dim = \(2d - g + 1\).

\[\Rightarrow \dim (\mathbb{P}^{(3g-3)}) = (2g + 2d - 2) - (2d - g + 1)\]
\[= 3g - 3.\]