Oct 14 : Moduli of curves

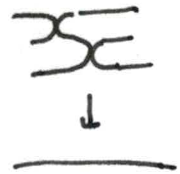
We work over $\mathbb{C}$ (not laziness – all arguments today will be topological.)

M_g = Coarse moduli space of smooth proj curves of genus g.

Thm : M_g is irreducible

Pf : We will construct an irreducible space that maps surjectively onto M_g.

$$H_{d,g} = \left\{ (C,f) \;\middle|\; \begin{array}{l} C \text{ is a smooth proj. curve} \\ \text{of genus } g,\ f: C \to \mathbb{P}^1 \text{ a} \\ \text{simply branched map of deg } d. \end{array} \right\} \Big/ \text{iso}$$



$\#$ branch points $= 2g + 2d - 2 =: b$

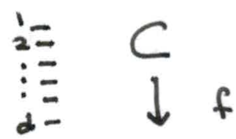
$$\begin{array}{ccccc} H_{d,g} & & (C,f) & & \\ \varphi \big\downarrow & & \big\downarrow & \text{and} & \\ \mathrm{Sym}^b(\mathbb{P}^1) \setminus \Delta & & br(f). & & \end{array} \qquad \begin{array}{ccc} H_{d,g} & \xrightarrow{\mu} & M_g \\ (C,f) & \longmapsto & C. \end{array}$$

Branched Covers : C, D compact Riemann Surfaces $f: C \to D$ finite map. of deg d.

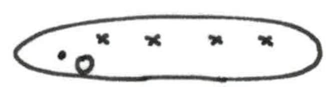
~~f_* $f^{-1}(D \setminus br f) \to D \setminus br f$ a covering space of deg d~~

Outside a finite set of points $B \subset D$, f is a covering space.

~~$f: f^{-1}$~~ $\quad f: C \setminus f^{-1}(B) \xrightarrow{d} D \setminus B$



Pick a base point $o \in D \setminus B$ and label $f^{-1}(o) = \{1, 2, \dots, d\}$. Lifting loops gives a homomorphism



D

$$\pi_1(D \setminus B, x) \xrightarrow{m} S_d .$$

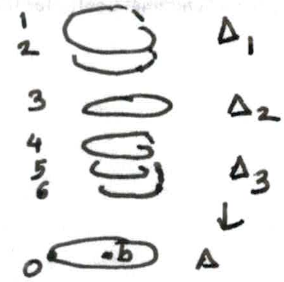
often called the monodromy representation. Changing the labelling by a permutation p changes m to pmp^{-1}. Conversely, any homomorphism $m: \pi_1(D \setminus B, x) \to S_d$ gives a covering space of deg d of $D \setminus B$. There is a unique way to complete this covering space to a deg d branched cover $C \to D$., where C is a compact R.S. Indeed, if $\Delta^* \subset D \setminus B$ is a punctured disk centered at $b \in B$, then

$$f^{-1}(\Delta^*) = \cancel{\Delta} \; \Delta^*_1 \sqcup \Delta^*_2 \sqcup \dots \sqcup \Delta^*_K \qquad \text{(union of punctured disks)}$$

with $f: \Delta^*_i \to \Delta^*$ of the form $z \mapsto z^{r_i}$ $\quad \Sigma r_i = d$

We complete this to

$$\Delta_1 \sqcup \Delta_2 \sqcup \dots \sqcup \Delta_k \to \Delta$$



monodromy = (12)(3)(456)

Note: In this picture a loop around b corresponds to the monodromy permutation

$$\underbrace{(\quad)}_{r_1\text{-cycle}} \underbrace{(\quad)}_{r_2\text{-cycle}} \dots \underbrace{(r_i \dots)}_{r_k\text{-cycle}}.$$

Rem: C is connected iff the image of m is a transitive subgroup of S_d.

Back to $\quad H_{d,g} \xrightarrow{\varphi} \mathrm{Sym}^b(\mathbb{P}^1) \setminus \Delta. \;=: \mathrm{Sym}^b(\mathbb{P}^1)^*$

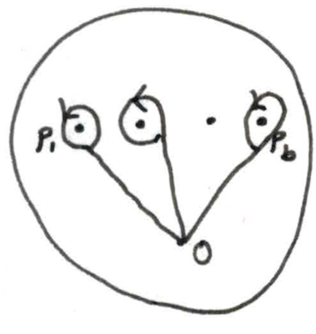
Fibers of φ: Over $(\mathbb{P}^1, \{P_1, \dots, P_b\})$

$$\|$$

$$\{ f : C \to \mathbb{P}^1 \text{ simply branched over } P_i \}$$

$$\|$$

$$\left\{ \begin{array}{l} m : \pi_1(\mathbb{P}^1 - P_1, \dots, P_b) \to S_d \text{ such that} \\ \text{① image is transitive} \\ \text{② a loop around } P_i \mapsto \text{transposition} \end{array} \right\} \Big/ \text{conj.}$$



$$\|$$

finite set indep of $P_1, \dots, P_b$. $\to$ $\left\{ (\sigma_1, \dots, \sigma_b) \;\middle|\; \begin{array}{l} \sigma_i \in S_d \text{ a simple transposition} \\ \prod \sigma_i = id \\ \sigma_i \text{ generate a transitive subgrp} \end{array} \right\} \Big/ \text{conjugation.}$

Use this to make $H_{d,g} \to \mathrm{Sym}^b(\mathbb{P}^1) \setminus \Delta$ a covering space.

Thus $H_{d,g}$ becomes a complex manifold (at least). The map to M_g is holomorphic. In fact $H_{d,g}$ is a quasi proj. variety — finite cover of a variety is a variety (Riemann Existence Thm).

Claim: $H_{d,g}$ is irreducible.

Pf: Enough to show $H_{d,g}$ is connected (because it is clearly smooth).
Equivalent to showing that the monodromy of the covering space

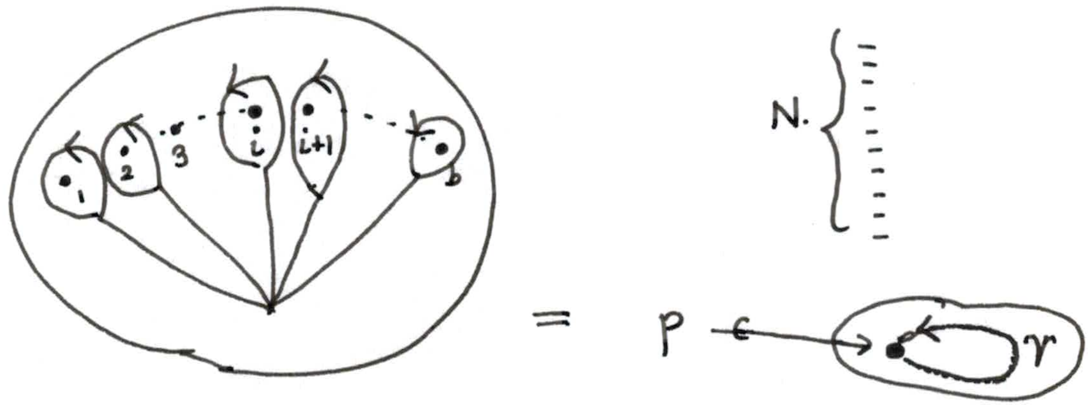
$$H_{d,g} \to \mathrm{Sym}^b(\mathbb{P}^1)^*$$

acts transitively on the fibers.

Fix a base point $p = (p_1, \dots, p_b) \in (\mathrm{Sym}^b \mathbb{P}^1)^*$. Fix loops as shown:

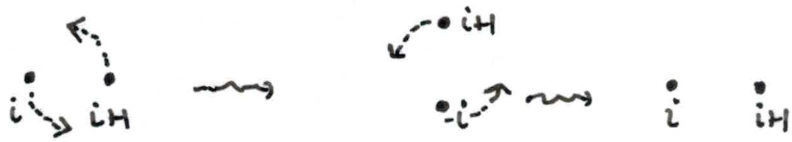
$$\varphi^{-1}(p) = \left\{ (\sigma_1, \dots, \sigma_b) \;\middle|\; \begin{array}{l} \sigma_i \text{ are transpositions, gen. trans. subgp} \\ \sigma_1 \sigma_2 \cdots \sigma_b = \mathrm{id} \end{array} \right\} \Big/ \text{conjugation}$$

$= \#N.$

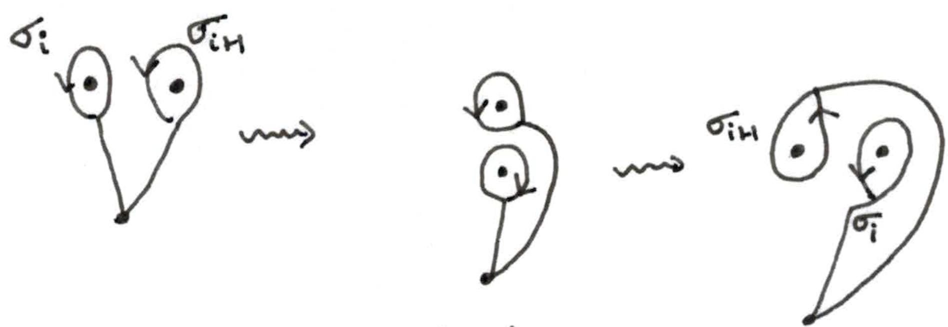


Take γ to be the loop that switches i & $i+1$. $(\mathrm{Sym}^b \mathbb{P}^1)^*$

as shown:



Trace the loops along to lift γ to $H_{d,g}$.



so, γ started at

$(\sigma_1, \dots, \sigma_i, \sigma_{i+1}, \dots, \sigma_b)$ $\rightsquigarrow$ ended at

in the std system of loops

$(\sigma_1, \dots, \sigma_i, \sigma_{i+1}, \dots, \sigma_b)$ in the non-standard system of loops

$\parallel$

$(\sigma_1, \dots, \sigma_i \sigma_{i+1} \sigma_i^{-1}, \sigma_i, \dots, \sigma_b)$ in the standard system.

Hence $\gamma : N \to N$ maps

$$(\sigma_1, \dots, \sigma_b) \mapsto (\sigma_1, \dots, \sigma_i \sigma_{i+1} \sigma_i^{-1}, \sigma_i, \dots, \sigma_b).$$

" Braid move."

Thm (Clebsch)

Any $(\sigma_1, \dots, \sigma_b) \in N$ can be brought into the form

$$(12)(12)\cdots(12)(23)(23)(34)(34)\cdots(d-1,d)(d-1,d).$$

by a sequence of braid moves. i.e. the monodromy of $H_{d,g} \to (\mathrm{Sym}^b \mathbb{P}^1)^*$ is transitive.

Now, for $d \gg 0$, the map $H_{d,g} \to M_g$ is surjective.

$\Rightarrow$ M_g is irreducible.

$\square$.

dim count: $\dim H_{d,g} = b = 2g+2d-2$

Fibers of $\mu: H_{d,g} \to M_g$ $\leftrightarrow$ $f: C \to \mathbb{P}^1$ for a fixed C.

① Choice of a line bundle of deg d on C $\rightarrow$ g

② Choice of two general sections of L $\rightarrow$ $2 \times (d-g+1) - 1$
so that $f = [s_1 : s_2]$
(up to scaling)

fiber dim $= 2d-g+1$.

$$\Rightarrow \dim(M_g) = (2g+2d-2) - (2d-g+1) = 3g-3.$$