Moduli of Curves - Nov 11

A few remarks about moduli spaces -

We have constructed $\mathcal{M}_g$ as a DM stack. We might also want a coarse moduli scheme M_g. There are two standard approaches.

① Keel-Mori thm ② GIT. — later.

Before describing them, we make a few def.

Def: $\mathfrak{X}/S$ is separated if $\Delta: \mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$ is proper.

Rem: Recall that if we have $T \to \mathfrak{X} \times \mathfrak{X}$ by (α, β) then

$$\begin{array}{ccc} \mathrm{Isom}(\alpha,\beta) & \longrightarrow & \mathfrak{X} \\ \downarrow & \square & \downarrow \Delta \\ T & \longrightarrow & \mathfrak{X} \times \mathfrak{X} \end{array}$$

So Δ proper $\Leftrightarrow$ Isom schemes are proper.

By the valuative criterion, this means the following. Let D be the spec of a DVR, and D^* the punctured spec. Then, given an iso $\alpha|_{D^*} \xrightarrow{\psi} \beta|_{D^*}$, it must extend to an iso $\alpha|_D \xrightarrow{\psi} \beta|_D$.

Check: $\mathcal{M}_g$ is separated (using the birat. geom. of surfaces).

Def: An algebraic space is ~~an étal~~ a sheaf in the étale topology ...... with an étale atlas (ie. A DM stack where the CFG is a sheaf)

eqv. a DM stack where Δ is an embedding.

~~(roughly)~~

Thm (Keel-Mori): Every separated DM stack $\mathfrak{X}$ has a coarse moduli space $\mathfrak{X} \to X$, where X is an algebraic space. (This map is initial among maps to alg spaces and big on geometric points.)

Ref: "Quotient by Groupoids". $R \rightrightarrows U \to X$

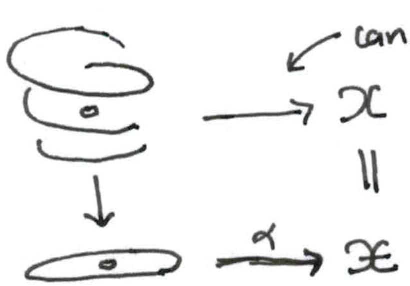
Def: $\mathfrak{X}/S$ is proper if $\exists\ Z/S$ proper and surj $Z \to \mathfrak{X}$.

A valuative criterion: Suppose $\mathfrak{X}/S$ is separated and finite type.

Let R be a DVR with fr. field K.

Then, given ~~specR~~ $\operatorname{spec} K \xrightarrow{\alpha} \mathfrak{X}$ $\exists$ finite separable extension K'/K s.t. $\operatorname{spec} K' \to \mathfrak{X}$ extends to $\operatorname{spec} R \to \mathfrak{X}$.

Picture



Ex. Let G be a finite group / spec k. Then BG is proper.

Val crit:


$\to BG \iff G$-bundle on the punct. disc.

Need not extend (in fact, will not extend if it is nontrivial).

But after a finite cover, it can be trivialized. $\Rightarrow$ extends.

(Keel-Mori) $\mathfrak{X}$ proper $\Rightarrow X$ proper.

GAGA: A proper algebraic space with an ample line bundle is ~~proj~~ ~~algebraic~~ a projective scheme.

Compact moduli stack $\overset{\text{Keel-Mori}}{\rightsquigarrow}$ compact coarse algebraic space $\underset{\text{ample line bundle}}{\overset{+}{\rightsquigarrow}}$ projective coarse moduli scheme.

For, Ample line bundle: Kleiman's criterion:

$L \to X$ is ample iff $L^r \cdot [Z] > 0$ $\forall\ r$ and $Z \subset X$ closed of dim r.

Furthermore - line bundles on $X \iff$ line bundles on $\mathfrak{X}$ (up to multiples).

Compactification of M_g.

Let k be an algebraically closed field.

Let C/k be a curve and $p \in C$ a k-point.

p is a node if $\hat{\mathcal{O}}_{C,p} \cong k[[x,y]]/xy$  "analytically".

Def: A nodal (or pre-stable) curve is a curve ~~with~~ C such that $\forall\, p \in C$, p is a smooth point or a node.

A stable curve is a (proper) pre-stable curve with finite automorphism group.

Ex.

NOT

Prop: Let $\tilde{C}$ be the normalization of a component of C.

Let $p_1, \dots, p_n \in \tilde{C}$ be the preimages of the nodes of C.

Then C is stable iff for every $\tilde{C}$, we have

$$2g(\tilde{C}) + n - 2 \geq 0$$

(i.e. genus 2 or higher, genus 1 with at least one special pt, genus 0 with at least 3 special points)

Def: $\overline{M}_g : \left\{ \begin{array}{c} C \\ \downarrow \pi \\ S \end{array} \;\middle|\; \begin{array}{l} \pi\text{- flat proper.} \\ \text{Geometric fibers are (connected) stable curves.} \end{array} \right\}$.

Thm: $\overline{M}_g$ is a Deligne-Mumford stack, smooth and proper over $\operatorname{spec} \mathbb{Z}$.

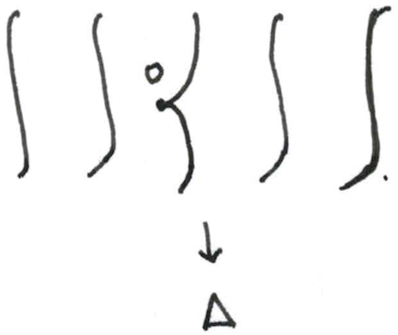
<u>Key Observation:</u> We can make the valuative criterion of properness work.

<u>Thm (Stable reduction)</u>: Let R be a DVR, K its fr. field.

Let $C \to \operatorname{spec} K$ be a stable curve. Then $\exists$ finite separable extension K'/K s.t. $C_{K'} \to K'$ extends to a stable curve $C_{R'} \to \operatorname{spec} R'$.

We'll prove this in char 0, where the proof is very direct and constructive.

<u>Example</u>: Plane curves acquiring a cusp.



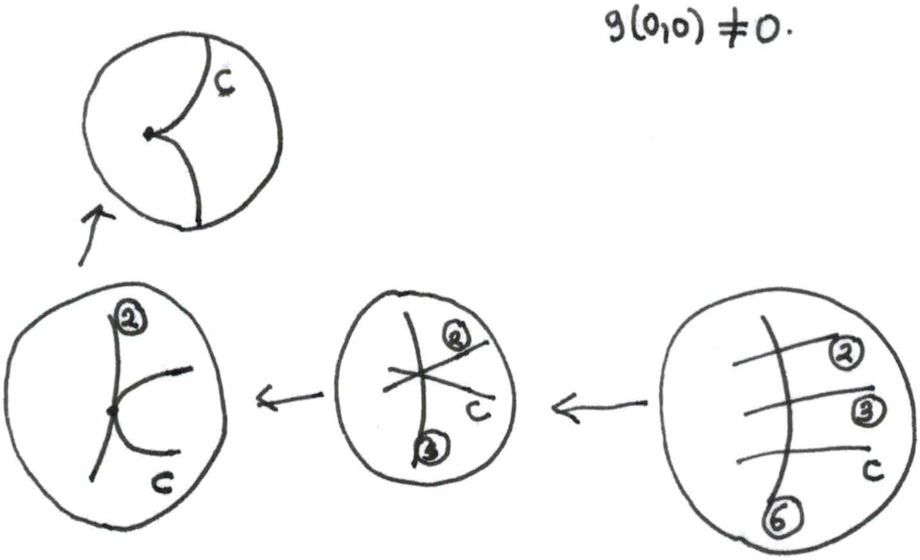
$\subset \mathbb{P}^2_x \Delta$.

$F + tG$

where F has a single cusp and G is general.
(i.e. G does not pass through the cusp of F and intersects F transversally.).

Locally near O: $(y^2 - x^3) + t\, g(x,y) \quad \subset \mathbb{C}[x,y,t]$.
$g(0,0) \neq 0$.

<u>Blow up</u> O:

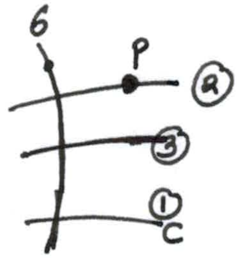


$$\begin{tikzcd} X''' \arrow[r, "\text{normalize}"] & X'' \arrow[r, "\alpha"] \arrow[d] & X' \arrow[d] \\ & \Delta \arrow[r, "t^2"'] & \Delta \end{tikzcd}$$

X' smooth.
$t=0$ is normal crossings.

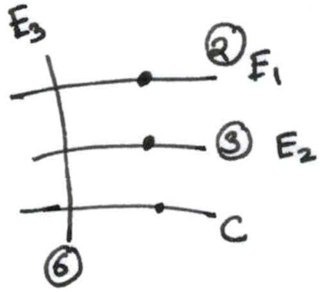
α unramified outside central fiber.

X''		X'
$\mathbb{C}[x,y,t]/(t^2-x^2)$	$\longrightarrow$	$\mathbb{C}[x,y,t]/(t-x^2)$
$\uparrow$ not smooth.		$\uparrow$ smooth.



$X''' = \mathbb{C}[x,y,t]/(t-x) \sqcup \mathbb{C}[x,y,t]/(t+x).$

$$\mathbb{C}[x,y,t]/(t^2-x^a) \longrightarrow \mathbb{C}[x,y,t]/(t-x^a)$$



a even

$\mathbb{C}[x,y,t]/(t-x^{a/2})$
$\sqcup$
$\mathbb{C}[x,y,t]/(t+x^{a/2})$

a odd

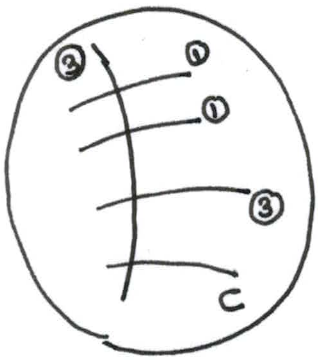
$\mathbb{C}[x,y,s]/(s^2-x).$

$s = \dfrac{t}{x^{\lfloor a/2 \rfloor}}$

i.e. $t = s\cdot x^{\lfloor a/2 \rfloor}.$

<u>Obs</u>: 1 $(X'')^\nu \longrightarrow X'$ branched double cover along

2 Preimage of E_3 has mult. 3 $\leftarrow$ one component.
E_1 has mult 1 $\leftarrow$ two components.
C has mult 1
E_2 has mult 3

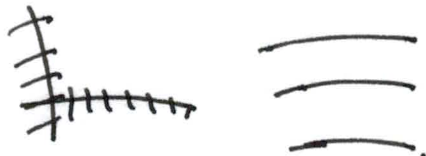
$(X'')^\nu =$ 

$\downarrow$

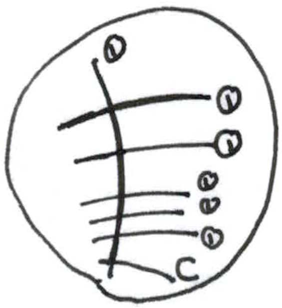
Δ

<u>Now</u>: Base change of order 3. and normalize.

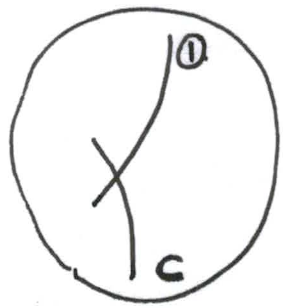
$(X''')^\nu \to (X'')^\nu$ ~~cyclic~~ triple cover branched along



$(X''')^\nu =$



$\xrightarrow{\text{contract -1 curves}}$



= Stable Reduction