Calculus III: Practice Final

Name: ____

Circle one: Section 6 Section 7 .

- Read the problems carefully.
- Show your work unless asked otherwise.
- Partial credit will be given for incomplete work.
- The exam contains 10 problems.
- The last page is the formula sheet, which you may detach.
- **Good luck!**

Question:	1	2	3	4	5	6	7	8	9	10	Total
Points:	10	10	10	10	10	10	10	10	10	10	100
Score:											

1. (10 points) Circle True or False. No justifation is needed.

(a) The curve traced by $\langle \cos^2(t), \sin^2(t) \rangle$ is a circle.

True False

(b) The plane $3x + 2y - z = 0$ is perpendicular to the line $x = 3t, y = 2t, z = -t$.

True False

(c) The function

$$f(x, y, z) = \begin{cases} \frac{\sin(x+y+z)}{x+y+z} & \text{if } x + y + z \neq 0 \\ 1 & \text{if } x + y + z = 0 \end{cases}$$

is continuous at $(0, 0, 0)$.

True False

(d) If the acceleration is constant, then the trajectory must be a straight line.

True False

(e) The complex number e^{2+3i} has magnitude 2.

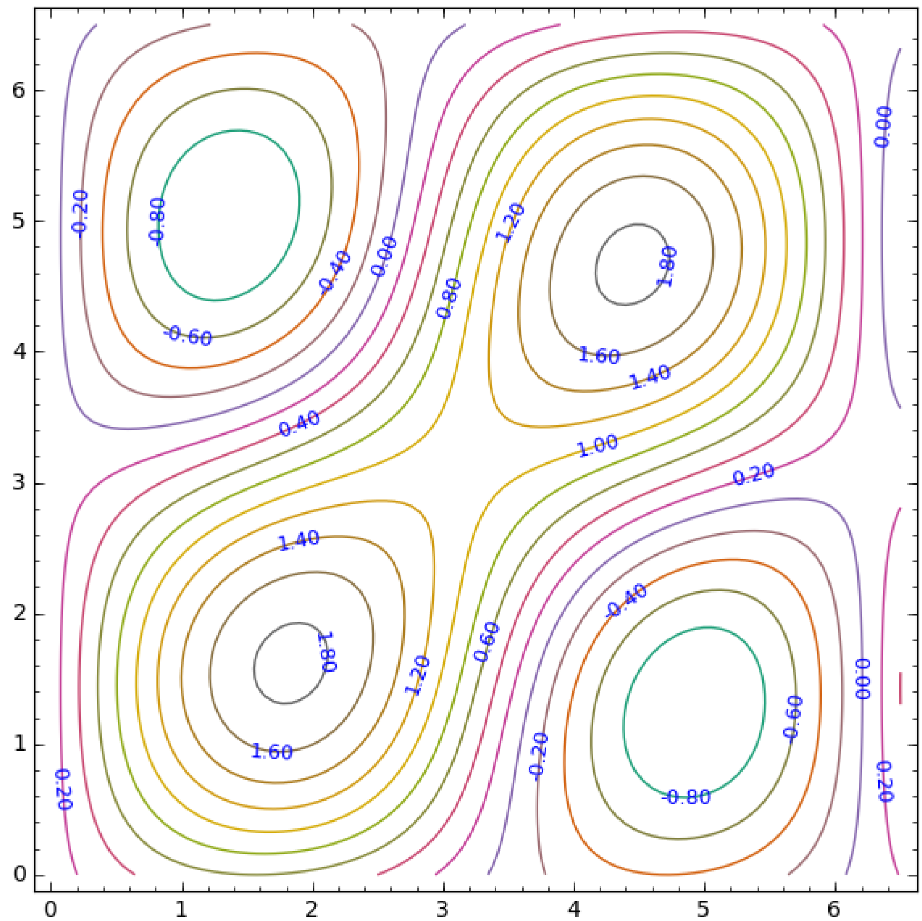
True False

2. In the following, compute $V \cdot W$, $V \times W$, and the cosine of the angle between V and W.

   (a) (5 points) $V = \langle 2, -1, 1 \rangle, \quad W = \langle 1, 3, -2 \rangle$.

   (b) (5 points) $V = \mathbf{i} + 3\mathbf{j}, \quad W = 3\mathbf{j} - 2\mathbf{k}$.

3. Use the contour plot of $f(x, y)$ to answer the questions. No justification is needed.



(a) (3 points) Mark any three critical points of f. Label them A, B, and C. Identify whether they are local minima, local maxima or saddle points.

(b) (2 points) Draw a vector at $(1, 1)$ indicating the direction of ∇f at $(1, 1)$.

(c) (3 points) Determine the sign of

1. $\frac{\partial f}{\partial x}(3, 4)$:
2. $\frac{\partial f}{\partial y}(2, 3)$:
3. $D_u f(5, 3)$ where u is the South–East direction:

(d) (2 points) Give a (admittedly rough) numerical estimate of $\frac{\partial f}{\partial x}(1, 1)$.

4. (a) (5 points) Write parametric equations for the tangent line at $\langle 1, 0, 1\rangle$ to the curve traced by $\langle t^2, \ln t, t^3\rangle$.

(b) (5 points) Write an equation of the normal plane to the curve at the same point.

5. (a) (5 points) Let $f(x, y) = xy/(x^2 + y^2)$. Find $\lim_{(x,y)\to(0,0)} f(x, y)$ or show that the limit does not exist.

(b) (5 points) Let $f(x, y) = \int_0^{xy} e^{t^2}\, dt$. Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

6. A ball of unit mass is thrown with the initial velocity of $\mathbf{i} + \mathbf{j}$. It experiences the force of gravity of magnitude 10 units in the $-\mathbf{j}$ direction and a force due to the wind of magnitude 1 unit in the $\mathbf{i}$ direction. Suppose the ball is initially at $(0, 4)$.

   (a) (7 points) Find the position of the ball at time t.

   (b) (3 points) Where is the ball when it hits the ground?

7. Suppose $u = e^{xy}$ where $x = st + s + t$ and $y = st - s - t$.

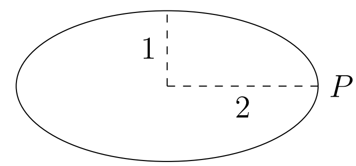
   (a) (2 points) Find the value of u when $s = 2$ and $t = 2$.

   (b) (8 points) Find an approximate numerical value of u when $s = 2.01$ and $t = 1.98$.

8. (10 points) Find all the critical points of the function $f(x,y) = x^3 + y^3 - 3xy$. Determine if they are local maxima, local minima or saddle points.

9. (10 points) A race track is in the shape of an ellipse with minor radius 1 km and major radius 2 km as shown. A car is going along this track at a constant speed of 100 km/h. Find the tangent and normal component of its accelaration when it as at P.

Race track



10. (10 points) You want to design a cylindrical cup that can hold 100π ml coffee. To minimize the material to be used, you decide to minimize the surface area. What is the radius and height of the optimal cup? (Ignore the thickness of the walls.)

LIST OF USEFUL IDENTITIES

1. Derivatives

(1) $\frac{d}{dx}x^n = nx^{n-1}$

(2) $\frac{d}{dx}\sin x = \cos x$

(3) $\frac{d}{dx}\cos x = -\sin x$

(4) $\frac{d}{dx}\tan x = \sec^2 x$

(5) $\frac{d}{dx}\cot x = -\csc^2 x$

(6) $\frac{d}{dx}\sec x = \sec x \tan x$

(7) $\frac{d}{dx}\csc x = -\csc x \cot x$

(8) $\frac{d}{dx}e^x = e^x$

(9) $\frac{d}{dx}\ln|x| = \frac{1}{x}$

(10) $\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}$

(11) $\frac{d}{dx}\arccos x = \frac{-1}{\sqrt{1-x^2}}$

(12) $\frac{d}{dx}\arctan x = \frac{1}{1+x^2}$

2. Trigonometry

(1) $\sin^2 x + \cos^2 x = 1$

(2) $\tan^2 x + 1 = \sec^2 x$

(3) $1 + \cot^2 x = \csc^2 x$

(4) $\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$

(5) $\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$

(6) $\sin^2 x = \frac{1-\cos 2x}{2}$

(7) $\cos^2 x = \frac{1+\cos 2x}{2}$.

3. Space curves

For a parametric space curve given by $\overline{r}(t)$

(1) Curvature $\quad \kappa = \dfrac{|r'(t) \times r''(t)|}{|r'(t)|^3}$.

(2) Tangent component of acceleration $\quad a_T = |r'(t)|' = \dfrac{r'(t) \cdot r''(t)}{|r'(t)|}$.

(3) Normal component of acceleration $\quad a_N = \kappa |r'(t)|^2 = \dfrac{|r'(t) \times r''(t)|}{|r'(t)|}$.