Calculus I: Practice Midterm I

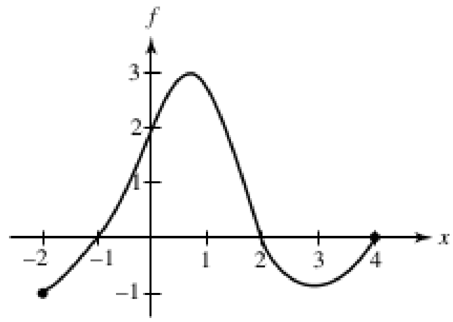
February 19, 2014

Name: ______________________________

- Write your solutions in the space provided. Continue on the back for more space.
- Show your work unless asked otherwise.
- Partial credit will be given for incomplete work.
- The exam contains 6 problems.
- **Good luck!**

Question	Points	Score
1	7	
2	8	
3	10	
4	9	
5	8	
6	8	
Total:	50	

1. Below is the graph of a function f.



(a) (3 points) Use the graph to (approximately) compute the following:

(a) $f(-1)$, $f(0)$, and $f(1)$.

Solution: $f(-1) = 0$, $f(0) = 2$, and $f(1) = 3$.

(b) All x such that $f(x) = 0$.

Solution: This is the set of x where the graph intersects the X-axis. These are -1, 2, and 4.

(c) The range of f.

Solution: The range of f is $[-1, 3]$.

(d) (4 points) Let $g(x) = x^2 + 1$. What is $f(g(1))$? What is $g(f(1))$?

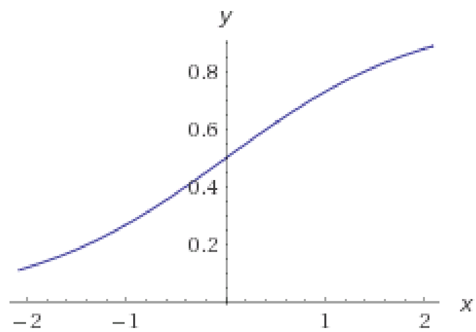
Solution:

$$f(g(1)) = f(2) = 0$$
$$g(f(1)) = g(3) = 3^2 + 1 = 10.$$

2. (8 points) Let

$$f(x) = \frac{e^x}{1+e^x}.$$

The graph of $f(x)$ is shown below



Does f have an inverse function? If yes, find a formula for $f^{-1}(y)$. If not, why not?

Solution: f has an inverse function because it is one-to-one (it satisfies the horizontal line test – each horizontal line intersects the graph in at most one point.)

To find a formula, let us write $y = f(x)$ and solve for x in terms of y. We have

$$\begin{aligned} y &= \frac{e^x}{1+e^x} \\ (1+e^x)y &= e^x \\ y + e^x y &= e^x \\ y &= e^x - e^x y \\ y &= (1-y)e^x \\ e^x &= \frac{y}{1-y} \\ x &= \ln\left(\frac{y}{1-y}\right) \\ x &= \ln y - \ln(1-y). \end{aligned}$$

So $f^{-1}(y) = \ln y - \ln(1-y)$.

3. Calculate each of the following limits, if it exists. Justify your answer.

(a) (3 points) $\lim_{t\to 0^+} e^{-10/t}$

Solution: As $t \to 0^+$, the quantity $x = -10/t$ becomes negative without bounds, that is, it approaches $-\infty$. So the given limit is equal to

$$\lim_{x\to -\infty} e^x.$$

From the graph of e^x, we know that this limit is 0.

(b) (3 points) $\lim_{x\to 5} \frac{x+10}{x-5}$

Solution: As $x \to 5$, the numerator approaches 15 but the denominator approaches 0. Therefore, $\lim_{x\to 5} \frac{x+10}{x-5}$ cannot exist.

If you are skeptical, here is a more rigorous argument. Suppose $\lim_{x\to 5} \frac{x+10}{x-5}$ existed and was equal to (a finite real number) L. Then by the product rule for limits

$$\begin{aligned}\lim_{x\to 5}(x+10) &= \lim_{x\to 5}(x-5)\cdot \lim_{x\to 5}\frac{x+10}{x-5}\\ &= 0\cdot L = 0.\end{aligned}$$

However, $\lim_{x\to 5}(x+10) = 15 \neq 0$. So our supposition was wrong.

(c) (4 points) $\lim_{x\to\infty} \frac{3x^2+10x-1}{x^2-5}$

Solution: We have

$$\begin{aligned}\lim_{x\to\infty}\frac{3x^2+10x-1}{x^2-5} &= \lim_{x\to\infty}\frac{x^2(3+10/x-1/x^2)}{x^2(1-5/x^2)}\\ &= \lim_{x\to\infty}\frac{3+10/x-1/x^2}{1-5/x^2}\\ &= \frac{\lim_{x\to\infty} 3+10/x-1/x^2}{\lim_{x\to\infty} 1-5/x^2}\\ &= \frac{3}{1} = 3.\end{aligned}$$

4. Let

$$h(x) = \begin{cases} |x-1|-1 & \text{for } x < 2 \\ 0 & \text{for } x = 2 \\ x^2 - 4 & \text{for } x > 2. \end{cases}.$$

(a) (3 points) Compute $\lim_{x\to 2^+} h(x)$.

Solution: We have

$$\begin{aligned} \lim_{x\to 2^+} h(x) &= \lim_{x\to 2^+} x^2 - 4 \\ &= 2^2 - 4 = 0 \end{aligned}$$

(b) (3 points) Compute $\lim_{x\to 2^-} h(x)$.

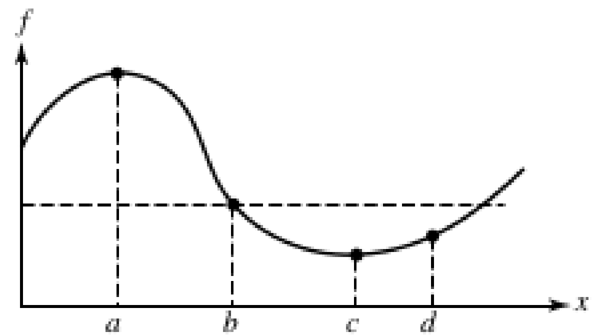
Solution: We have

$$\begin{aligned} \lim_{x\to 2^-} h(x) &= \lim_{x\to 2^-} (|x-1|-1) \\ &= \lim_{x\to 2^-} |x-1|-1 = 0. \end{aligned}$$

(c) (3 points) Is $h(x)$ continuous at 2?

Solution: Since the right hand limit is equal to the left hand limit, $\lim_{x\to 2} h(x)$ exists and is equal to 0. Since we also have $h(2) = 0$, the function is continuous.

5. (a) (4 points) Suppose $f(x)$ is given by the following graph



Using the graph, put the following in ascending order

$$0, \quad f'(d), \quad \frac{f(c)-f(b)}{c-b}, \quad f'(b).$$

Solution: Use that $f'(d)$ is the slope of the tangent line at $(d, f(d))$ etc. and $\frac{f(c)-f(b)}{c-b}$ is the slope of the secant line through $(c, f(c))$ and $(b, f(b))$. By looking at these slopes, we get

$$f'(b) < \frac{f(c)-f(b)}{c-b} < 0 < f'(d).$$

(b) (4 points) Suppose $g(x)$ is given by the formula

$$g(x) = 2x^3 - 3x + 4.$$

Compute $g(1)$ and $g'(1)$. Use this to find an approximate value of $g(1.1)$.

Solution: By using the rules for derivatives, we get

$$g'(x) = 6x^2 - 3$$
$$g''(x) = 12x.$$

Substituting $x = 1$, we get $g'(1) = 3$ and $g''(1) = 12$. We also have $g(1) = 3$. So

$$g(1.1) - g(1) \approx (1.1 - 1)g'(1) = 0.1 \cdot 3 = 0.3$$

So $g(1.1) \approx g(1) + 0.3 = 3.3$.

6. Let

$$f(x) = \frac{3x}{1+x}.$$

(a) (6 points) Use the definition of the derivative to find $f'(2)$.

Solution: We use the definition:

$$\begin{aligned} f'(2) &= \lim_{h\to 0} \frac{f(2+h) - f(2)}{h} \\ &= \lim_{h\to 0} \frac{\frac{3(2+h)}{1+2+h} - \frac{6}{3}}{h} \\ &= \lim_{h\to 0} \frac{\frac{6+3h}{3+h} - 2}{h} \\ &= \lim_{h\to 0} \frac{6+3h-6-2h}{h(3+h)} \\ &= \lim_{h\to 0} \frac{h}{h(3+h)} \\ &= \lim_{h\to 0} \frac{1}{3+h} \\ &= \frac{1}{3}. \end{aligned}$$

(b) (2 points) Is f increasing or decreasing at $x = 2$?

Solution: $f'(2) > 0$ means that f is increasing at $x = 2$.