10. $\int_0^T (x^4 - 8x + 7)\, dx = \left[\frac{1}{5}x^5 - 4x^2 + 7x\right]_0^T = \left(\frac{1}{5}T^5 - 4T^2 + 7T\right) - 0 = \frac{1}{5}T^5 - 4T^2 + 7T$

12. Let $u = 1 - x$, so $du = -dx$ and $dx = -du$. When $x = 0$, $u = 1$; when $x = 1$, $u = 0$. Thus,

$\int_0^1 (1-x)^9\, dx = \int_1^0 u^9(-du) = \int_0^1 u^9\, du = \frac{1}{10}\left[u^{10}\right]_0^1 = \frac{1}{10}(1-0) = \frac{1}{10}$.

43. $F(x) = \int_0^x \frac{t^2}{1+t^3}\, dt \quad \Rightarrow \quad F'(x) = \frac{d}{dx}\int_0^x \frac{t^2}{1+t^3}\, dt = \frac{x^2}{1+x^3}$

46. Let $u = \sin x$. Then $\frac{du}{dx} = \cos x$. Also, $\frac{dg}{dx} = \frac{dg}{du}\frac{du}{dx}$, so

$$g'(x) = \frac{d}{dx}\int_1^{\sin x} \frac{1-t^2}{1+t^4}\, dt = \frac{d}{du}\int_1^u \frac{1-t^2}{1+t^4}\, dt \cdot \frac{du}{dx} = \frac{1-u^2}{1+u^4}\cdot\frac{du}{dx} = \frac{1-\sin^2 x}{1+\sin^4 x}\cdot \cos x = \frac{\cos^3 x}{1+\sin^4 x}$$

70. $\lim_{n\to\infty} \frac{1}{n}\left[\left(\frac{1}{n}\right)^9 + \left(\frac{2}{n}\right)^9 + \left(\frac{3}{n}\right)^9 + \cdots + \left(\frac{n}{n}\right)^9\right] = \lim_{n\to\infty} \frac{1-0}{n}\sum_{i=1}^{n}\left(\frac{i}{n}\right)^9 = \int_0^1 x^9\, dx = \left[\frac{x^{10}}{10}\right]_0^1 = \frac{1}{10}$

The limit is based on Riemann sums using right endpoints and subintervals of equal length.

1. $A = \int_{x=0}^{x=4} (y_T - y_B)\, dx = \int_0^4 \left[(5x - x^2) - x\right] dx = \int_0^4 (4x - x^2)\, dx = \left[2x^2 - \frac{1}{3}x^3\right]_0^4 = \left(32 - \frac{64}{3}\right) - (0) = \frac{32}{3}$

2. $A = \int_0^2 \left(\sqrt{x+2} - \frac{1}{x+1}\right) dx = \left[\frac{2}{3}(x+2)^{3/2} - \ln(x+1)\right]_0^2 = \left[\frac{2}{3}(4)^{3/2} - \ln 3\right] - \left[\frac{2}{3}(2)^{3/2} - \ln 1\right]$

$= \frac{16}{3} - \ln 3 - \frac{4}{3}\sqrt{2}$

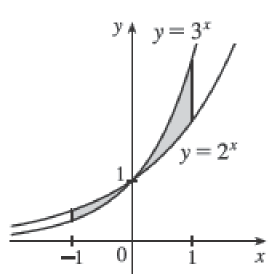
4. $A = \int_0^3 \left[(2y - y^2) - (y^2 - 4y)\right] dy = \int_0^3 (-2y^2 + 6y)\, dy = \left[-\frac{2}{3}y^3 + 3y^2\right]_0^3 = (-18 + 27) - 0 = 9$

32. $A = \int_{-1}^{1} |3^x - 2^x|\, dx = \int_{-1}^{0} (2^x - 3^x)\, dx + \int_0^1 (3^x - 2^x)\, dx$

$= \left[\frac{2^x}{\ln^2} - \frac{3^x}{\ln^3}\right]_{-1}^{0} + \left[\frac{3^x}{\ln 3} - \frac{2^x}{\ln 2}\right]_0^1$

$= \left(\frac{1}{\ln 2} - \frac{1}{\ln 3}\right) - \left(\frac{1}{2\ln 2} - \frac{1}{3\ln 3}\right) + \left(\frac{3}{\ln 3} - \frac{2}{\ln 2}\right) - \left(\frac{1}{\ln 3} - \frac{1}{\ln 2}\right)$

$= \frac{2 - 1 - 4 + 2}{2\ln 2} + \frac{-3 + 1 + 9 - 3}{3\ln 3} = \frac{4}{3\ln 3} - \frac{1}{2\ln 2}$



43. 1 second $= \frac{1}{3600}$ hour, so 10 s $= \frac{1}{360}$ h. With the given data, we can take $n = 5$ to use the Midpoint Rule.

$\Delta t = \frac{1/360-0}{5} = \frac{1}{1800}$, so

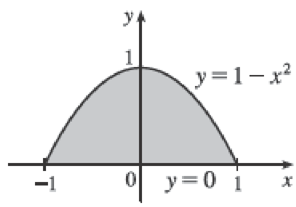
$$\begin{aligned}\text{distance}_{\text{Kelly}} - \text{distance}_{\text{Chris}} &= \int_0^{1/360} v_K\,dt - \int_0^{1/360} v_C\,dt = \int_0^{1/360} (v_K - v_C)\,dt \\ &\approx M_5 = \tfrac{1}{1800}\left[(v_K - v_C)(1) + (v_K - v_C)(3) + (v_K - v_C)(5)\right. \\ &\qquad \left.+ (v_K - v_C)(7) + (v_K - v_C)(9)\right] \\ &= \tfrac{1}{1800}[(22-20) + (52-46) + (71-62) + (86-75) + (98-86)] \\ &= \tfrac{1}{1800}(2+6+9+11+12) = \tfrac{1}{1800}(40) = \tfrac{1}{45} \text{ mile, or } 117\tfrac{1}{3} \text{ feet}\end{aligned}$$

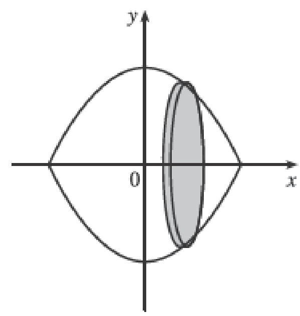
44. If x = distance from left end of pool and $w = w(x)$ = width at x, then the Midpoint Rule with $n = 4$ and

$\Delta x = \dfrac{b-a}{n} = \dfrac{8 \cdot 2 - 0}{4} = 4$ gives Area $= \int_0^{16} w\,dx \approx 4(6.2 + 6.8 + 5.0 + 4.8) = 4(22.8) = 91.2\ \text{m}^2$.

2. A cross-section is a disk with radius $1 - x^2$, so its area is $A(x) = \pi(1-x^2)^2$.

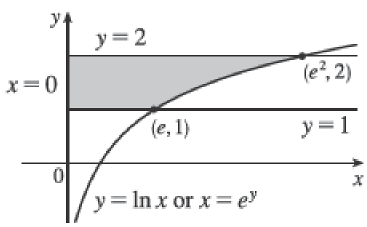
$$\begin{aligned}V &= \int_{-1}^{1} A(x)\,dx = \int_{-1}^{1} \pi(1-x^2)^2\,dx \\ &= 2\pi \int_0^1 (1 - 2x^2 + x^4)\,dx = 2\pi\left[x - \tfrac{2}{3}x^3 + \tfrac{1}{5}x^5\right]_0^1 \\ &= 2\pi\left(1 - \tfrac{2}{3} + \tfrac{1}{5}\right) = 2\pi\left(\tfrac{8}{15}\right) = \tfrac{16}{15}\pi\end{aligned}$$

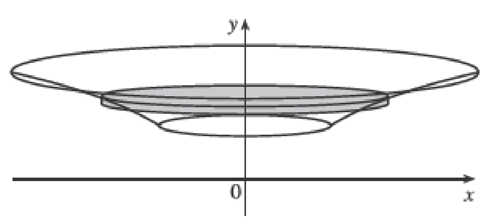




6. A cross-section is a disk with radius e^y [since $y = \ln x$], so its area is $A(y) = \pi(e^y)^2$.

$$V = \int_1^2 \pi(e^y)^2\,dy = \pi\int_1^2 e^{2y}\,dy = \pi\left[\frac{1}{2}e^{2y}\right]_1^2 = \frac{\pi}{2}\left(e^4 - e^2\right)$$





19. $\mathscr{R}_1$ about OA (the line $y = 0$):

$$V = \int_0^1 A(x)\,dx = \int_0^1 \pi(x)^2\,dx = \pi\left[\tfrac{1}{3}x^3\right]_0^1 = \tfrac{1}{3}\pi$$

21. $\mathscr{R}_1$ about AB (the line $x = 1$):

$$V = \int_0^1 A(y)\,dy = \int_0^1 \pi(1-y)^2\,dy = \pi\int_0^1 (1 - 2y + y^2)\,dy = \pi\left[y - y^2 + \tfrac{1}{3}y^3\right]_0^1 = \tfrac{1}{3}\pi$$

40. $\pi \int_{-1}^{1} (1 - y^2)^2 \, dy$ describes the volume of the solid obtained by rotating the region

$\mathcal{R} = \{(x, y) \mid -1 \le y \le 1, 0 \le x \le 1 - y^2\}$ of the xy-plane about the y-axis.

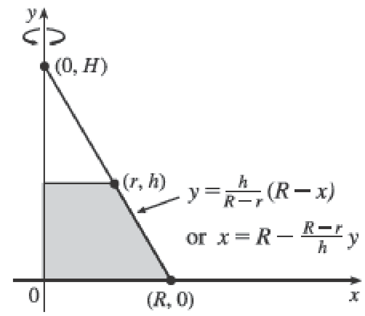
48. $V = \pi \int_0^h \left(R - \frac{R - r}{h} y\right)^2 dy$

$= \pi \int_0^h \left[R^2 - \frac{2R(R - r)}{h} y + \left(\frac{R - r}{h}\right)^2 y^2\right] dy$

$= \pi \left[R^2 y - \frac{R(R - r)}{h} y^2 + \frac{1}{3}\left(\frac{R - r}{h}\right)^2 y^3\right]_0^h$

$= \pi\left[R^2 h - R(R - r)h + \frac{1}{3}(R - r)^2 h\right]$

$= \frac{1}{3}\pi h\left[3Rr + (R^2 - 2Rr + r^2)\right] = \frac{1}{3}\pi h(R^2 + Rr + r^2)$



Another solution: $\frac{H}{R} = \frac{H - h}{r}$ by similar triangles. Therefore, $Hr = HR - hR \;\Rightarrow\; hR = H(R - r) \;\Rightarrow$

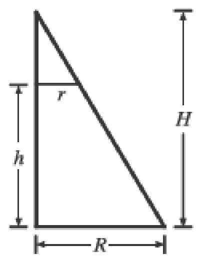
$H = \frac{hR}{R - r}$. Now

$V = \frac{1}{3}\pi R^2 H - \frac{1}{3}\pi r^2 (H - h)$ [by Exercise 49]

$= \frac{1}{3}\pi R^2 \frac{hR}{R - r} - \frac{1}{3}\pi r^2 \frac{rh}{R - r}$ $\left[H - h = \frac{rH}{R} = \frac{rhR}{R(R - r)}\right]$

$= \frac{1}{3}\pi h \frac{R^3 - r^3}{R - r} = \frac{1}{3}\pi h\left(R^2 + Rr + r^2\right)$

$= \frac{1}{3}\left[\pi R^2 + \pi r^2 + \sqrt{(\pi R^2)(\pi r^2)}\right] h = \frac{1}{3}\left(A_1 + A_2 + \sqrt{A_1 A_2}\right) h$



where A_1 and A_2 are the areas of the bases of the frustum. (See Exercise 50 for a related result.)