Calculus 1: Practice Midterm 2

April 2, 2015

Name: ______________________________

- Write your solutions in the space provided. Continue on the back for more space.
- Show your work. Just writing the final answer will receive little credit.
- Partial credit will be given for incomplete work.
- The exam contains 6 problems.
- The last three pages are left blank for scratch work. You may detach them.
- **Good luck!**

1. Compute the following.

(a) $f'(x)$ where $f(x) = \sin(2x)\ln(x)$.

Solution: By the product and chain rules:

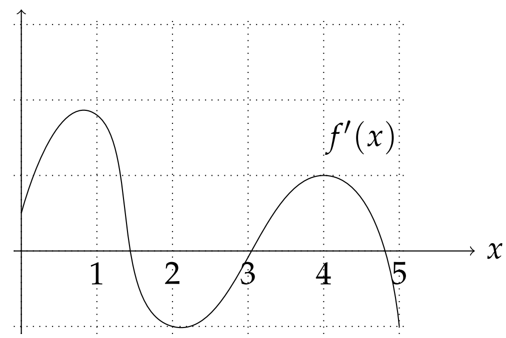
$$f'(x) = 2\cos(2x)\ln(x) + \frac{\sin(2x)}{x}.$$

(b) $f''(1)$ where $f(x) = e^{3x} + \frac{1}{x}$.

Solution:

$$\begin{aligned} f'(x) &= 3e^{3x} - x^{-2} \\ f''(x) &= 9e^{3x} + 2x^{-3} \\ f''(1) &= 9e^3 + 2. \end{aligned}$$

2. Below is the graph of the derivative $f'(x)$ of a function $f(x)$ defined on $(0,5)$.



(a) What are the critical points of $f(x)$? Which are local minima and which are local maxima?

Solution: The critical points of $f(x)$ are those where $f'(x) = 0$. These are $x \approx 1.5$, $x = 3$, and $x \approx 4.8$.

	0 to 1.5	1.5 to 3	4 to 4.8	4.8 to 5
Sign of f'	+	-	+	-
Direction of f	Inc	Dec	Inc	Dec.

Therefore 1.5 and 4.8 are local maxima and $x = 3$ is a local minimum.

(b) Find the inflection points of the graph of $f(x)$.

Solution: The inflection points of $f(x)$ are those where $f'(x)$ changes direction (which are the same as the points where $f''(x)$ changes sign). They are $x \approx 0.9$, $x \approx 2$, and $x \approx 4$.

(c) Find an x (any x) at which the graph of $f(x)$ is increasing and concave down.

Solution: Any point between 1 and 1.5 or 4 and 4.8 would work.

3. Calculate
$$\lim_{x\to+\infty}\left(1-\frac{1}{x}\right)^x.$$

Solution: Let
$$L=\lim_{x\to+\infty}\left(1-\frac{1}{x}\right)^x.$$
Then
$$\begin{aligned}\ln L &= \lim_{x\to+\infty} x\ln(1-1/x)\\ &= \lim_{x\to+\infty}\frac{\ln(1-1/x)}{1/x}.\end{aligned}$$
As $x\to+\infty$, we have $1/x\to 0$ and $\ln(1-1/x)\to\ln(1)=0$. So we have a limit of the form "0/0." Applying l'Hôpital's rule:
$$\begin{aligned}\ln L &= \lim_{x\to+\infty}\frac{\frac{1}{1-1/x}\times\frac{1}{x^2}}{\frac{-1}{x^2}}\\ &= \frac{-1}{1-1/x}\\ &= \frac{-1}{1}\\ &= -1.\end{aligned}$$
Since $\ln L=-1$, we get $L=e^{-1}=1/e$.

4. (10 points) The *kinetic energy* of an object is given by the formula

$$K = \frac{1}{2}mv^2,$$

where m is its mass and is v its velocity. The standard unit for K is joules, for m is kilograms, and for v is meters per second.

(a) Suppose a rocket of mass 3×10^6 kilograms is moving at the speed of 5×10^3 meters per second and is generating kinetic energy at the rate of 60×10^{11} joules per second. At that moment, what the rate of change of its velocity?

Solution: We have the equation

$$K = \frac{1}{2}mv^2.$$

The mass $m = 3 \times 10^6$ is constant. Differentiating with respect to time t, we get

$$\frac{dK}{dt} = mv\frac{dv}{dt}.$$

We are given that $dK/dt = 60 \times 10^{11}$ and $v = 5 \times 10^3$. So we get

$$\begin{aligned}\frac{dv}{dt} &= \frac{1}{mv}\frac{dK}{dt} \\ &= \frac{1}{3 \times 10^6 \times 5 \times 10^3} \times 60 \times 10^{11} \\ &= \frac{1}{15 \times 10^9} \times 60 \times 10^{11} \\ &= 400.\end{aligned}$$

So the velocity is changing at the rate of 400 m/s^2.

(b) Use linear approximation to estimate the velocity after 2 seconds.

Solution:

$$\begin{aligned}\text{Velocity after 2 seconds} &\approx \text{Current velocity} + \text{Rate of change of velocity} \times 2 \\ &= 5 \times 10^3 + 800 \\ &= 5800\ m/s^2.\end{aligned}$$

5. We want to calculate $\sqrt[3]{4}$ using Newton's method. Write a function whose root is $\sqrt[3]{4}$, and execute the first two steps of Newton's method starting with the initial guess $x_0 = 1$.

Solution: $\sqrt[3]{4}$ is a root of $x^3 - 4$. Let $f(x) = x^3 - 4$. So $f'(x) = 3x^2$.

We know that the steps in Newton's method are computed by

$$\begin{aligned} x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} \\ &= x_n - \frac{x_n^3 - 4}{3x_n^2}. \end{aligned}$$

Starting with $x_0 = 1$, we get

$$\begin{aligned} x_1 &= 1 - \frac{-3}{3} = 2 \\ x_2 &= 2 - \frac{4}{12} = 2 - 1/3 = 5/3. \end{aligned}$$

So we get an approximation $\sqrt[3]{4} \approx 5/3$.

6. Suppose Coca Cola were to design its cylindrical Coke can so that it held 100π ml soda and used the minimum amount of metal. What would be the the radius of the optimum can?

Solution: Let the radius be r and height be h. Then the metal used is given by the total surface area

$$2\pi(rh + r^2)$$

We want the volume to be 100π. So

$$100\pi = \pi r^2 h.$$

We thus get

$$h = 100/r^2.$$

Substituting in the surface area, we get

$$2\pi(100/r + r^2).$$

We want to minimize this function for $r \in (0, +\infty)$. We may as well minimize

$$f(r) = (100/r + r^2).$$

The critical points are found by setting $f'(r) = 0$. We get

$$\begin{aligned} f'(r) = -100/r^2 + 2r &= 0 \\ 2r &= 100/r^2 \\ r^3 &= 50. \end{aligned}$$

So $r = \sqrt[3]{50}$ is the only critical point. For $r < \sqrt[3]{50}$, the derivative is negative (plug in $r = 1$ in $f'(r)$). For $r > \sqrt[3]{50}$, the derivative is positive (plug in $r = 10$ or $r =$ a trillion in $f'(r)$). So the function decreases from 0 to $\sqrt[3]{50}$ and increases from $\sqrt[3]{50}$ to $+\infty$. Therefore, it has a global minimum at $\sqrt[3]{50}$.

So the optimal radius is $\sqrt[3]{50}$.

(It comes out to be about 3.6 cm. The actual radius of a coke can is 3.25 cm. Shows that their design choices are not guided solely by the amount of metal used.)