Calculus 1: Practice Final

May 6, 2015

Name: Solutions

- Write your solutions in the space provided. Continue on the back for more space.
- Show your work unless asked otherwise.
- Partial credit will be given for incomplete work.
- The exam contains 10 problems.
- **Good luck!**

1. Let
$$f(x) = xe^{-x}.$$

(a) Find $f'(x)$.

$$f'(x) = 1 \cdot e^{-x} + x e^{-x} \cdot (-1)$$
$$= e^{-x} - x e^{-x}$$

(b) Find $f''(x)$.

$$f''(x) = e^{-x} \cdot (-1) - (x e^{-x})'$$
$$= -e^{-x} - e^{-x} + x e^{-x}$$
$$= x e^{-x} - 2 e^{-x}$$

(c) Is $f(x)$ concave up or concave down at $x = 1$?

$f(x)$ concave up / down $\leftrightarrow$ $f''(x)$ positive / negative.

$$f''(1) = e^{-1} - 2e^{-1} = -e^{-1} = \frac{-1}{e} < 0$$

$\Rightarrow$ $f(x)$ is concave down at $x=1$.

2. Let

$$f(x) = x\sqrt{4-x^2}.$$

(a) Find the domain of f.

For $f(x)$ to be well-defined, we must have

$$4 - x^2 \geqslant 0$$

$$\Leftrightarrow \quad 4 \geqslant x^2$$

$$\Leftrightarrow \quad 2 \geqslant x \geqslant -2$$

So the domain is $[-2, 2]$

(b) Find the global minima and maxima of f.

We first find the critical points. For those, we want

$$f'(x) = 0$$

$$f'(x) = 1 \cdot \sqrt{4-x^2} + x \cdot \frac{1}{2}(4-x^2)^{-\frac{1}{2}} \cdot (-2x)$$

$$= \sqrt{4-x^2} - \frac{x^2}{\sqrt{4-x^2}}$$

So $f'(x) = 0$ means $\sqrt{4-x^2} = \dfrac{x^2}{\sqrt{4-x^2}}$

$$\Leftrightarrow \quad 4 - x^2 = x^2$$

$$\Leftrightarrow \quad 4 = 2x^2 \quad \Leftrightarrow \quad x^2 = 2 \quad \Leftrightarrow \quad x = \pm\sqrt{2}.$$

To check global max/min, we compare the values of $f(x)$ at the end points and at the critical points.

$f(-2) = 0$ $\qquad f(\sqrt{2}) = \sqrt{2} \cdot \sqrt{2} = 2$

$f(2) = 0$ $\qquad f(-\sqrt{2}) = -\sqrt{2} \cdot \sqrt{2} = -2$

So $\sqrt{2}$ is global max, $-\sqrt{2}$ is global min.

3. Let
$$f(x)=\frac{x^2+1}{x^2-1}.$$
Find the horizontal and vertical asymptotes of the graph of $f(x)$.

Horizontal :

The line $y=a$ is a horizontal asymptote if
$$\lim_{x\to\pm\infty} f(x)=a.$$

We have

We divided Num. & Den. by the dominant power of x.
$$\lim_{x\to+\infty}\frac{x^2+1}{x^2-1}=\lim_{x\to+\infty}\frac{1+1/x^2}{1-1/x^2}=\frac{1}{1}=1$$
$$\lim_{x\to-\infty}\frac{x^2+1}{x^2-1}=\lim_{x\to-\infty}\frac{1+1/x^2}{1-1/x^2}=\frac{1}{1}=1.$$

So $y=1$ is a horizontal asymptote

Vertical : $x=a$ is a vertical asymptote if
$$\lim_{x\to a^\pm} f(x)=\pm\infty.$$

This is only possible if $\lim_{x\to a^\pm}(x^2-1)=0$ (i.e. denom. must go to 0.)

So $a^2-1=0 \Rightarrow a=-1$ or $+1$.

For both $a=-1$, and $a=1$, the numerator of $f(x)$ goes to a nonzero number (2), so

$$\lim_{x\to 1^\pm} f(x)=\lim_{x\to -1^\pm} f(x)=\pm\infty \Rightarrow \left.\begin{array}{l}x=1\\x=-1\end{array}\right\}\text{ Vertical asymptotes}$$

4. Evaluate the following

(a) $f'(x)$ where $f(x) = x^{\sin x}$.

Set $y = x^{\sin x}$

$\ln y = \sin x \ln x$ Diff wrt x

$\frac{y'}{y} = \cos x \ln x + \frac{\sin x}{x}$

So $y' = y\left(\cos x \ln x + \frac{\sin x}{x}\right)$

$= x^{\sin x}\left(\cos x \ln x + \frac{\sin x}{x}\right)$

(b) $\lim_{x\to 0} \frac{\sin(3x)\cos(4x)}{\sin(5x)}$

As $x \to 0$, Numerator $\to \sin(0).\cos(0) = 0$

Denominator $\to \sin(0) = 0$.

So we can apply L'Hôpital's rule:

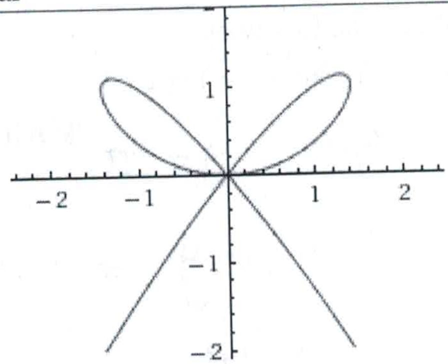
Given limit

$= \lim_{x\to 0} \frac{3\cos(3x)\cos(4x) - 4\sin(3x)\sin(4x)}{5\cos(5x)}$

$= \frac{3\cdot 1\cdot 1 - 4\cdot 0}{5\cdot 1}$

$= \frac{3}{5}$

5. The "bow curve" shown here is defined by the equation $x^4 = 3x^2y - 2y^3$. Find the equation of the tangent line to the curve at the point $(1, 1)$.



We need to find $\frac{dy}{dx}$ at $(1,1)$.

We use implicit differentiation.

$$x^4 = 3x^2y - 2y^3 \qquad \text{diff wrt } x$$

$$4x^3 = 6xy + 3x^2\frac{dy}{dx} - 6y^2\frac{dy}{dx}$$

Set $x=y=1$.

$$4 = 6 + 3\frac{dy}{dx} - 6\frac{dy}{dx}$$

$$-2 = -3\frac{dy}{dx} \qquad \text{so} \qquad \frac{dy}{dx} = \frac{2}{3}.$$

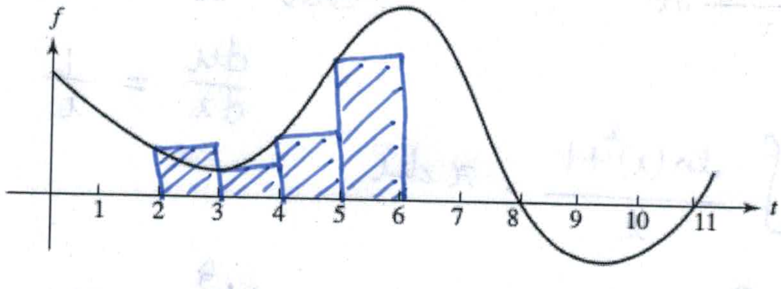
So the tangent line has slope $\frac{2}{3}$ & passes through (1,1

Equation: $\frac{y-1}{x-1} = \frac{2}{3}$

so $3y - 3 = 2x - 2$

i.e. $3y - 2x = 1$.

6. The following is the graph of a function $f(x)$.



(a) Write (but do not evaluate) the Riemann sum for the integral $\int_2^6 f(x)dx$ using 4 parts and left end-points. Draw on the graph the area that the sum represents.

$\Delta x = \frac{6-2}{4} = 1.$ $\quad x_0 = 2 \quad x_1 = 3 \quad x_2 = 4 \quad x_3 = 5 \quad x_4 = 6$

Riemann sum $= f(x_0)\cdot\Delta x + f(x_1)\cdot\Delta x + f(x_2)\Delta x + f(x_3)\Delta x$

$= f(2) + f(3) + f(4) + f(5)$

(b) Consider the new function $F(t)$ defined by the formula

$$F(t) = \int_2^{t^2} f(x)\, dx.$$

Determine the sign (positive/negative) of the following quantities:

1. $F(1)$ $= \int_2^1 f(x)\,dx = -\int_1^2 f(x)\,dx$ ← Negative
2. $F'(2)$ $= 4 \cdot f(4)$ ← Positive
3. $F''(3)$ $= 36 f'(9) + 2f(9)$ ← Negative

By the FTC & chain rule

$F'(t) = f(t^2)\cdot 2t.$ Then

$F''(t) = f'(t^2)\cdot 2t \cdot 2t + f(t^2)\cdot 2$

$= 4t^2 f'(t^2) + 2f(t^2)$

7. Evaluate the integrals

(a) $\displaystyle\int \frac{\ln(x)^2+1}{x}\,dx$

Set $u = \ln x$

$\frac{du}{dx} = \frac{1}{x}$, $dx = x\,du$

$= \displaystyle\int \frac{\ln(x)^2+1}{x} \cdot x\,du$

$= \displaystyle\int (u^2+1)\,du = \frac{u^3}{3} + u + C$

$= \frac{\ln(x)^3}{3} + \ln x + C.$

(b) $\displaystyle\int_0^1 \frac{x}{1+x^2}\,dx$

Use $u = 1+x^2$

$\frac{du}{dx} = 2x$

$dx = \frac{du}{2x}$

$= \displaystyle\int_{u=1}^{u=2} \frac{x}{1+x^2} \cdot \frac{du}{2x}$

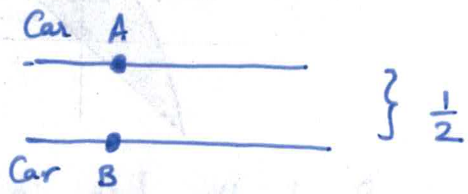
$= \frac{1}{2}\displaystyle\int_1^2 \frac{du}{u} = \frac{1}{2}(\ln u)\Big|_1^2$

$= \frac{1}{2}\ln(2).$

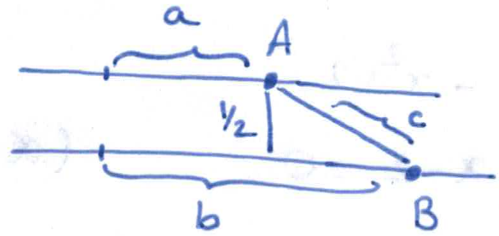
8. Two cars start side by side on parallel roads that are 0.5 miles apart. The first car travels at 30 mph and the second at 40 mph. After one hour, what is the rate of change of the distance between the two cars?

Picture

$t=0$:



t :



At $t=1$: $a = 30$ $\quad \frac{da}{dt} = 30$

$b = 40$ $\quad \frac{db}{dt} = 40$

Always : $(b-a)^2 + \left(\frac{1}{2}\right)^2 = c^2$ diff. wrt time t

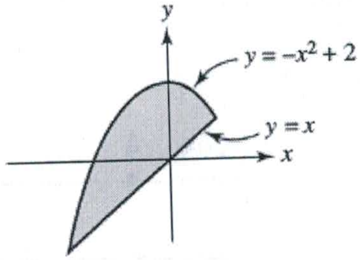
$$2(b-a)\left(\frac{db}{dt} - \frac{da}{dt}\right) = 2c\,\frac{dc}{dt}$$

At $t=1$: $c = \sqrt{100 + 1/4}$.

So $2\cdot(10)(10) = 2\sqrt{100.25}\,\frac{dc}{dt}$

So $\frac{dc}{dt} = \frac{100}{\sqrt{100.25}}$ at $t=1$.

9. Find the shaded (unsigned) area.



Let us first find the two end points.

$$x = -x^2+2$$

$$x^2+x-2=0 \qquad (x-1)(x+2)$$

so $x=-2$ & $x=1$

$$\text{Area} = \int_{-2}^{1} (\text{bigger function} - \text{smaller function})\, dx$$

$$= \int_{-2}^{1} (-x^2+2-x)\, dx$$

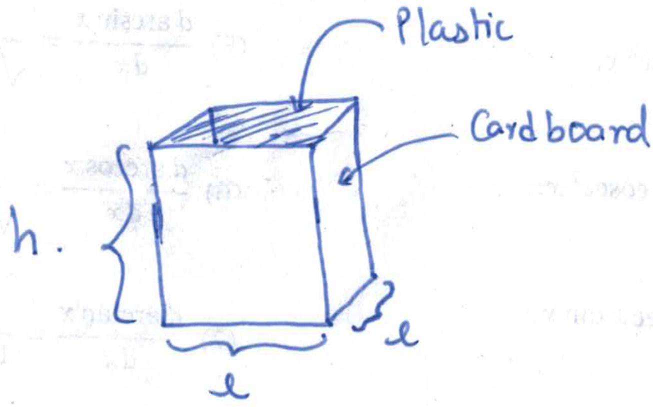
$$= -\frac{x^3}{3}+2x-\frac{x^2}{2}\Big|_{-2}^{1}$$

$$= \left(\frac{-1}{3}+2-\frac{1}{2}\right)-\left(\frac{8}{3}-4-2\right)$$

$$= \frac{7}{6}-\left(\frac{-10}{3}\right) = \frac{27}{6} = \frac{9}{2}$$

10. You are designing a box for blueberries which has a volume of 250 cubic centimeters and a square base. It is made of cardboard except that it has a see-through plastic top. Suppose the plastic costs three times as much as the cardboard. What are the dimensions of the box that minimize the total cost of the materials?

Picture :



$V = l^2 h = 250.$

Cost = 3 × Top area + Area of other sides & bottom.

$= 3l^2 + \underbrace{l^2}_{\text{bottom}} + \underbrace{4hl}_{\text{4 sides}}$

$= 4l^2 + 4hl$ $\qquad h = \dfrac{250}{l^2}$

$= 4l^2 + \dfrac{1000}{l}$ ← minimize. Domain $(0, +\infty)$.

Diff. & set $=0$

$0 = 8l - \dfrac{1000}{l^2}$ so $8l = \dfrac{1000}{l^2} \Rightarrow l^3 = 125$

$\Rightarrow l = 5$

Signs of the derivative : $(0 \xrightarrow{\quad -ve \quad} 5 \xrightarrow{\quad +ve \quad})$.

Direction of cost function : ↘ ↗ so $l=5$ is global min

Min cost for $l = 5$ & $h = \dfrac{250}{25} = 10.$

FORMULA SHEET

1. Derivatives

(1) $\dfrac{d\tan x}{dx} = \sec^2 x.$

(2) $\dfrac{d\cot x}{dx} = -\operatorname{cosec}^2 x.$

(3) $\dfrac{d\sec x}{dx} = \sec x \tan x.$

(4) $\dfrac{d\operatorname{cosec} x}{dx} = -\operatorname{cosec} x \cot x.$

(5) $\dfrac{d\arcsin x}{dx} = \dfrac{1}{\sqrt{1-x^2}}.$

(6) $\dfrac{d\arccos x}{dx} = \dfrac{-1}{\sqrt{1-x^2}}.$

(7) $\dfrac{d\arctan x}{dx} = \dfrac{1}{1+x^2}.$

2. Surface Areas and volumes

(1) Sphere of radius r:

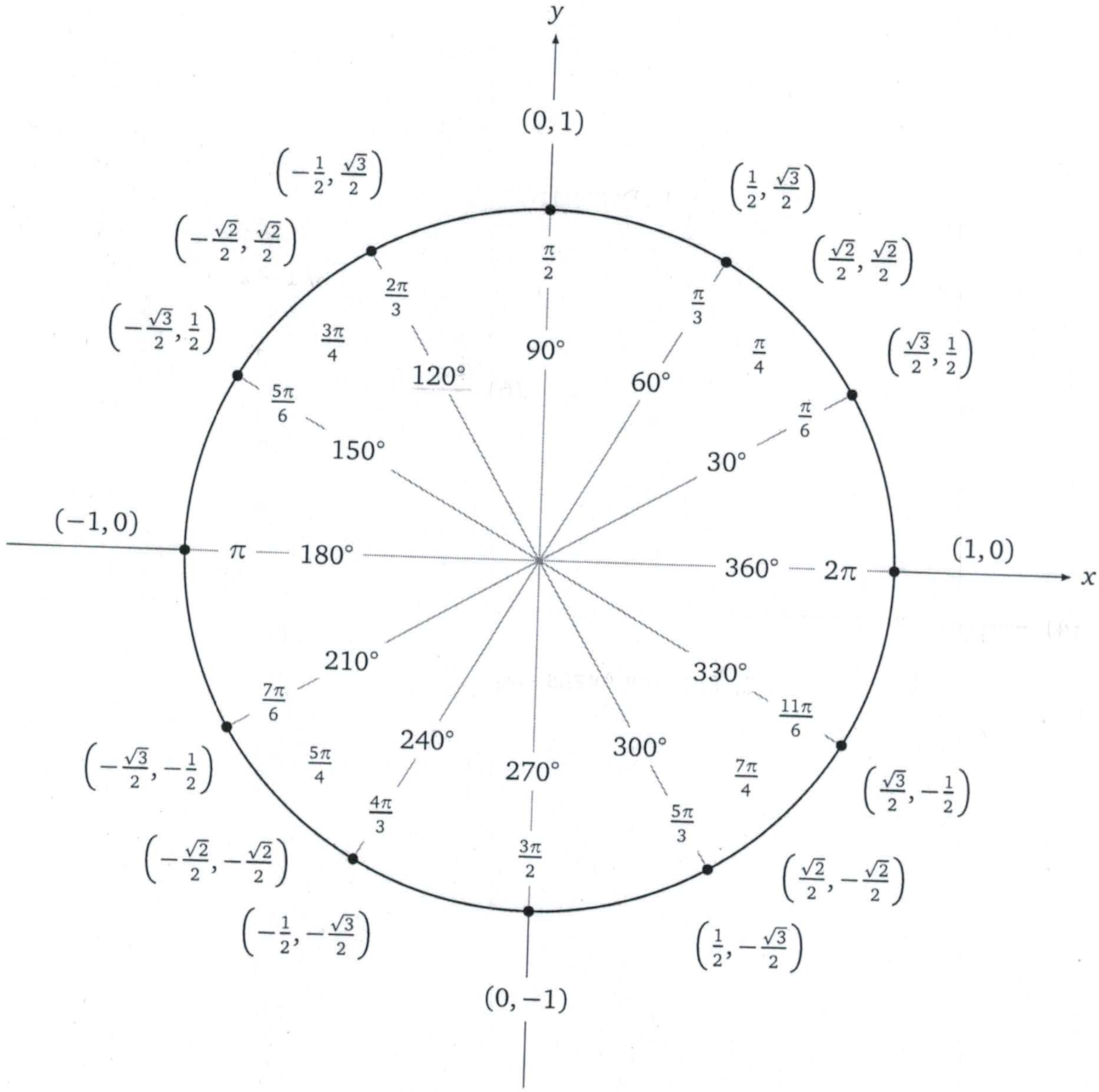
- Volume $= \dfrac{4}{3}\pi r^3$,
- Surface area $= 4\pi r^2$.

(2) Cylinder of radius r and height h:

- Volume $= \pi r^2 h$,
- Curved surface area $= 2\pi r h$,
- Total surface area $= 2\pi r h + 2\pi r^2$.

(3) Cone of radius r and height h:

- Volume $= \dfrac{1}{3}\pi r^2 h$,
- Curved surface area $= \pi r\sqrt{r^2+h^2}$,
- Total surface area $= \pi r\sqrt{r^2+h^2} + \pi r^2$.



y
x
$(0,1)$
$\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$
$\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$
$\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$
$\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$
$\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$
$\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$
$(-1,0)$
$(1,0)$
$\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$
$\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$
$\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$
$\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$
$\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$
$\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$
$(0,-1)$
$\frac{\pi}{2}$ 90°
$\frac{2\pi}{3}$ 120°
$\frac{3\pi}{4}$
$\frac{5\pi}{6}$ 150°
π 180°
$\frac{7\pi}{6}$ 210°
$\frac{5\pi}{4}$
$\frac{4\pi}{3}$ 240°
$\frac{3\pi}{2}$ 270°
$\frac{5\pi}{3}$ 300°
$\frac{7\pi}{4}$
$\frac{11\pi}{6}$ 330°
360° 2π
$\frac{\pi}{6}$ 30°
$\frac{\pi}{4}$
$\frac{\pi}{3}$ 60°