Calculus I: Practice Final

May 4, 2015

Name: ______________________________

- Write your solutions in the space provided. Continue on the back for more space.
- Show your work unless asked otherwise.
- Partial credit will be given for incomplete work.
- The exam contains 10 problems.
- **Good luck!**

1. Let
$$f(x) = xe^{-x}.$$

(a) Find $f'(x)$.

Solution: By the product and chain rules,
$$\begin{aligned} f'(x) &= 1 \cdot e^{-x} - xe^{-x} \\ &= e^{-x} - xe^{-x}. \end{aligned}$$

(b) Is $f(x)$ concave up or concave down at $x = 1$?

Solution: We must find the second derivative at $x = 1$. Differentiating $f'(x)$ once more, we get
$$\begin{aligned} f''(x) &= -e^{-x} - (xe^{-x})' \\ &= -e^{-x} - e^{-x} + xe^{-x} \\ &= e^{-x}(x-2). \end{aligned}$$

Therefore, $f''(1) = e^{-1}(-1) < 0$. So, $f(x)$ is concave down at $x = 1$.

2. Let
$$f(x) = x\sqrt{4-x^2}.$$

(a) Find the domain of f.

Solution: $f(x)$ is well-defined when $x^2 \leq 4$, that is $-2 \leq x \leq 2$. So the domain is $[-2,2]$.

(b) Find the global minima and maxima of f.

Solution: We first find the critical points. We have

$$\begin{aligned} f'(x) &= (4-x^2)^{1/2} - x \cdot 2x \cdot \frac{1}{2}(4-x^2)^{-1/2}. \\ &= \sqrt{4-x^2} - \frac{x^2}{\sqrt{4-x^2}} \\ &= \frac{4-2x^2}{\sqrt{4-x^2}} \end{aligned}$$

This is zero when $x = \pm\sqrt{2}$. The global minima/maxima are either at the endpoints (± 2)or at the critical points ($\pm\sqrt{2}$). We check the four values

$$f(-2) = 0, f(2) = 0, f(\sqrt{2}) = 2, f(-\sqrt{2}) = -2.$$

Thus, the global maximum is at $x = \sqrt{2}$ and the minimum is at $x = -\sqrt{2}$.

3. Evaluate the following

(a) $f'(x)$ for $f(x) = e^{3x}\sin(4x)$

Solution:

$$\begin{aligned} f'(x) &= 3e^{3x}\sin(4x) + e^{3x}4\cos(4x) \\ &= e^{3x}(3\sin(4x) + 4\cos(4x)). \end{aligned}$$

(b) $\int_0^{\pi/2} \sin(x)\cos(x)dx.$

Solution: Let $u = \sin(x)$. Then $\cos(x)dx = du$. Therefore

$$\begin{aligned} \int_0^{\pi/2} \sin(x)\cos(x)dx &= \int_0^1 udu \\ &= \left.\frac{x^2}{2}\right|_0^1 \\ &= \frac{1}{2}. \end{aligned}$$

(c) $\lim_{x\to 0} \frac{\sin(3x)\cos(4x)}{\sin(5x)}$

Solution: Both the numerator and the denominator go to 0 as x goes to 0. By L'Hôpital,

$$\begin{aligned} \lim_{x\to 0} \frac{\sin(3x)\cos(4x)}{\sin(5x)} &= \lim_{x\to 0} \frac{3\cos(3x)\cos(4x) - 4\sin(3x)\sin(4x)}{5\cos(5x)} \\ &= \frac{3}{5}. \end{aligned}$$

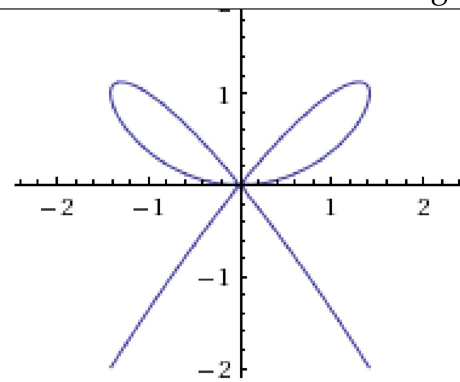
4. Find $G'(x)$ for the function $G(x)$ defined by

$$G(x) = \int_0^{\sqrt{x}} \sin(t^2)dt.$$

Solution: Let $F(x) = \int_0^x \sin(t^2)dt$. By the Fundamental Theorem of Calculus, $F'(x) = \sin(x^2)$. Also $G(x) = F(\sqrt{x})$. By the chain rule,

$$\begin{aligned} G'(x) &= F'(\sqrt{x}) \cdot \frac{1}{2}x^{-1/2} \\ &= \frac{\sin(x)}{2\sqrt{x}}. \end{aligned}$$

5. The "bow curve" shown here is defined by the equation $x^4 = 3x^2y - 2y^3$. Find the equation of the tangent line to the curve at the point $(1, 1)$.



Solution: We first find the slope of the tangent line at $(1, 1)$. By differentiating the given equation, with respect to x, we get

$$x^4 = 3x^2y - 2y^3$$
$$4x^3 = 6xy + 3x^2\frac{dy}{dx} - 6y^2\frac{dy}{dx}.$$

Substituting $x = 1, y = 1$ gives

$$4 = 6 - 3\frac{dy}{dx}$$
$$\frac{dy}{dx} = \frac{2}{3}.$$

Threfore, the slope of the tangent line is $2/3$. Since the tangent line passes through $(1, 1)$, its equation is

$$\frac{y-1}{x-1} = \frac{2}{3}$$
$$3y - 3 = 2x - 2$$
$$3y - 2x = 1.$$

6. (a) Let $f(x) = x\ln(x) - x$. Find $f'(x)$.

Solution: By the product rule,

$$\begin{aligned} f'(x) &= 1 \cdot \ln(x) + x \cdot \frac{1}{x} - 1 \\ &= \ln(x). \end{aligned}$$

(b) Use the previous part to calculate $\int_1^e \ln(x) dx$.

Solution: By the Fundamental Theorem of Calculus,

$$\begin{aligned} \int_1^e \ln(x) dx &= f(x)|_1^e \\ &= e\ln(e) - e - (1\ln(1) - 1) \\ &= 1. \end{aligned}$$

7. The velocity of a particle at time t is given by $v(t) = 4t^3 - 3t^2$. Suppose the particle is at $x = 0$ at $t = 0$. Where is the particle at $t = 2$?

Solution: Let $p(t)$ be the position of the particle at time t. What we are given is

$$p'(t) = 4t^3 - 3t^2,$$

and $p(0) = 0$. We get

$$\begin{aligned} p(t) &= \int (4t^3 - 3t^2)dt \\ &= t^4 - t^3 + C. \end{aligned}$$

Since $p(0) = 0$, we get

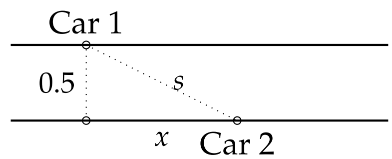
$$0 = 0 + C,$$

that is $C = 0$. Therefore $p(t) = t^4 - t^3$. Putting $t = 2$, we get

$$p(2) = 8.$$

8. Two cars are on parallel roads that are 0.5 miles apart. They start side by side, but the first car travels at 30 mph and the second at 40 mph. After one hour, what is the rate of change of the distance between the two cars?

Solution: Here is a picture of the situation



By Pythagoras's theorem,

$$s^2 = x^2 + 0.5^2.$$

By differentiating with respect to t, we get

$$2s\frac{ds}{dt} = 2x\frac{dx}{dt}.$$

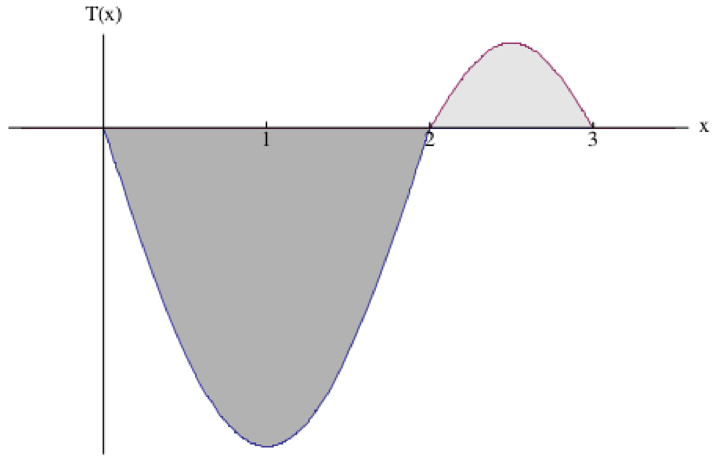
Therefore,

$$\frac{ds}{dt} = \frac{x}{s}\frac{dx}{dt}.$$

We know that $x = 10$, therefore $s = \sqrt{10^2 + 0.5^2}$. Also $\frac{dx}{dt}$ is the difference between the speeds, which is 10. Therefore,

$$\frac{ds}{dt} = \frac{10}{\sqrt{100.25}} \cdot 10 = \frac{100}{\sqrt{100.25}}.$$

9. The following figure shows the graph of a function $T(x)$. The region with the darker shading has area 15 and the region with the lighter shading has area 2.



Compute the following:

(a) $\int_0^3 T(x)dx$

Solution:

$$\int_0^3 T(x)dx = \int_0^2 T(x)dx + \int_2^3 T(x)dx = -15 + 2 = -13.$$

(b) $\int_0^2 (T(x) + x)dx$

Solution:

$$\begin{aligned}\int_0^2 (T(x) + x)dx &= \int_0^2 T(x)dx + \int_0^2 xdx \\ &= -15 + \left.\frac{x^2}{2}\right|_0^2 \\ &= -15 + 2 = -13.\end{aligned}$$

10. You are designing a cardboard box for blueberries. It has to be a cuboid with a square cardboard base, cardboard sides, and a see through plastic top. The volume of the box must be 250 cubic centimeters. Suppose the plastic costs three times as much as the cardboard. What are the dimensions of the box that meet the specifications and minimize the cost? Recall that the volume of a cuboid is the product of the length, width, and height.

Solution: Suppose the base of the box has length and width x and the height is h. Then the volume is

$$V = x^2 h.$$

The cost of the materials is

$$C = x^2(\text{bottom}) + 4xh(\text{four sides}) + 3x^2(\text{top}) = 4x^2 + 4xh.$$

Since the volume is supposed to be 250, we have

$$250 = x^2 h \implies h = \frac{250}{x^2}.$$

Substituting in C, we get

$$C = 4x^2 + \frac{1000}{x}.$$

We want to minimize C as x ranges over $(0, \infty)$.

We first find the critical points.

$$\begin{aligned} \frac{dC}{dx} &= 8x - \frac{1000}{x^2} = 0 \\ \implies 8x &= \frac{1000}{x^2} \\ \implies x^3 &= \frac{1000}{8} \\ \implies x &= \frac{10}{2} = 5. \end{aligned}$$

Therefore, the only critical point is $x = 5$. Next, we must make sure that this is the global minimum. For $x < 5$, the sign of $\frac{dC}{dx}$ is negative (plug in $x = 1$, for example). For $x > 5$, the sign of $\frac{dC}{dx}$ is positive (plug in $x =$ a billion, for example). Therefore, C is decreasing for $x < 5$ and increasing for $x > 5$. It follows that C has a global minimum at $x = 5$.

So the cost is minimized when the base is 5×5 and the height is 10.