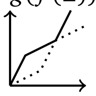
CALCULUS 1: MIDTERM 1 SOLUTIONS

(1) (a) $f(1) = 0.5$ and $\lim_{x\to 3} f(x) = 3$.
(b) Domain $= [0,5]$, range $= [0,4]$
(c) $g(f(2)) = 3$ and $f(g(2)) = 4$.
(d) 

(2) (a) $f(x) = 0.5x + 2.5$.
(b) Let $y = f(x)$. So

$$y = 0.5x + 2.5$$
$$x = (y - 2.5)/0.5 = 2y - 5.$$

Thus, $f^{-1}(x) = 2x - 5$.
(c) If the taxi goes at 30 miles per hour for t hours, then the distance covered is $x = 30t$. Substituting in $f(x)$ we get the fare $15t + 2.5$.

(3) (a) We have

$$\lim_{x\to 0+} h(x) = \lim_{x\to 0+} e^{1-x^5} = e$$
$$\lim_{x\to 0-} h(x) = \lim_{x\to 0+} \arctan(1-x^5) = \arctan(1) = \pi/4$$

.

Since $e \neq \pi/4$, the one sided limits are unequal. Therefore the limit $\lim_{x\to 0} h(x)$ does not exist.
(b) To find the horizontal asymptotes, we must find the limit of $h(x)$ as $x \to +\infty$ and $x \to -\infty$. We have

$$\lim_{x\to +\infty} h(x) = \lim_{x\to +\infty} e^{1-x^5} = \lim_{t\to -\infty} e^t = 0,$$

where we let $t = 1 - x^5$ which goes to $-\infty$ as x goes to $+\infty$. Similarly,

$$\lim_{x\to -\infty} h(x) = \lim_{x\to +\infty} \arctan(1-x^5) = \lim_{t\to +\infty} \arctan(t) = \pi/2,$$

where we let $t = 1 - x^5$ which goes to $+\infty$ as x goes to $-\infty$.
Thus, we get the horizontal asymptotes $y = 0$ and $y = \pi/2$.

(4) (a)

$$\lim_{x\to +\infty} \frac{\sqrt{3x^4+1}}{x^2+x+1} = \lim_{x\to +\infty} \frac{\sqrt{3x^4+1}/x^2}{(x^2+x+1)/x^2} = \lim_{x\to +\infty} \frac{\sqrt{(3x^4+1)/x^4}}{(x^2+x+1)/x^2} = \lim_{x\to +\infty} \frac{\sqrt{3+1/x^4}}{1+1/x+1/x^2} = \frac{\sqrt{3+0}}{1+0+0} = \sqrt{3}.$$

(b)

$$\lim_{x\to 0}\left(\frac{1}{x^2-x} + \frac{1}{x^2+x}\right) = \lim_{x\to 0}\left(\frac{1}{x(x-1)} + \frac{1}{x(x+1)}\right) = \lim_{x\to 0}\left(\frac{(x+1)+(x-1)}{x(x-1)(x+1)}\right) = \lim_{x\to 0}\left(\frac{2x}{x(x-1)(x+1)}\right) = \lim_{x\to 0}\left(\frac{2}{(x-1)(x+1)}\right) = \frac{2}{(0-1)(0+1)} = -2.$$

(c) We know that

$$-1 \le \sin(1/x) \le 1.$$

Multiplying by $\ln(1+x)$ for $x > 0$, we get

$$-\ln(1+x) \le \ln(1+x)\sin(1/x) \le \ln(1+x).$$

Since

$$\lim_{x\to 0+} -\ln(1+x) = \lim_{x\to 0+} \ln(1+x) = \ln(1) = 0,$$

the squeeze theorem tells us that

$$\lim_{x\to 0^+} \ln(1+x)\sin(1/x) = 0.$$

(d)

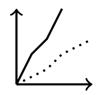
$$\lim_{x\to 0} \frac{e^{2x}-1}{e^x-1} = \lim_{x\to 0} \frac{(e^x-1)(e^x+1)}{e^x-1} = \lim_{x\to 0}(e^x+1) = e^0 + 1 = 2.$$

(5) (a) From the definition of the derivative,

$$f'(3) = \lim_{x\to 3} \frac{f(x)-f(3)}{x-3} = \lim_{x\to 3} \frac{\frac{x+1}{x-1} - \frac{4}{2}}{x-3} = \lim_{x\to 3} \frac{\frac{x+1}{x-1} - 2}{x-3} = \lim_{x\to 3} \frac{\frac{x+1-2x+2}{x-1}}{x-3} = \lim_{x\to 3} \frac{3-x}{(x-1)(x-3)} = \lim_{x\to 3} \frac{-1}{(x-1)} = \frac{-1}{2}.$$

(b) Since $f'(3) < 0$, the tangent point downwards and hence $f(x)$ is decreasing near $x = 3$.

(1) (a) $f(3) = 2$ and $\lim_{x\to 4} f(x) = 2.5$.
(b) Domain $= [0, 5]$, range $= [0, 3]$
(c) $g(f(2)) = 3$ and $f(g(2)) = 3$.
(d)



(2) (a) $f(x) = 0.5x + 3.5$.
(b) Let $y = f(x)$. So

$$y = 0.5x + 3.5$$
$$x = (y - 3.5)/0.5 = 2y - 7.$$

Thus, $f^{-1}(x) = 2x - 7$.
(c) If the taxi goes at 20 miles per hour for t hours, then the distance covered is $x = 20t$. Substituting in $f(x)$ we get the fare $10t + 3.5$.

(3) (a) We have

$$\lim_{x\to 0+} h(x) = \lim_{x\to 0+} e^{2-x^3} = e^2$$
$$\lim_{x\to 0-} h(x) = \lim_{x\to 0+} \arctan(2 - x^3) = \arctan(2) \approx 1.1$$

.

Since $e^2 \neq \arctan(2)$, the one sided limits are unequal. Therefore the limit $\lim_{x\to 0} h(x)$ does not exist.
(b) To find the horizontal asymptotes, we must find the limit of $h(x)$ as $x \to +\infty$ and $x \to -\infty$. We have

$$\lim_{x\to+\infty} h(x) = \lim_{x\to+\infty} e^{2-x^3} = \lim_{t\to-\infty} e^t = 0,$$

where we let $t = 2 - x^3$ which goes to $-\infty$ as x goes to $+\infty$. Similarly,

$$\lim_{x\to-\infty} h(x) = \lim_{x\to+\infty} \arctan(2 - x^3) = \lim_{t\to+\infty} \arctan(t) = \pi/2,$$

where we let $t = 2 - x^3$ which goes to $+\infty$ as x goes to $-\infty$.
Thus, we get the horizontal asymptotes $y = 0$ and $y = \pi/2$.

(4) (a)

$$\lim_{x\to+\infty} \frac{\sqrt{x^4+1}}{3x^2+x+1} = \lim_{x\to+\infty} \frac{\sqrt{x^4+1}/x^2}{(3x^2+x+1)/x^2} = \lim_{x\to+\infty} \frac{\sqrt{(x^4+1)/x^4}}{(3x^2+x+1)/x^2} = \lim_{x\to+\infty} \frac{\sqrt{1+1/x^4}}{3+1/x+1/x^2} = \frac{\sqrt{1+0}}{3+0+0} = \frac{1}{3}$$

(b) We know that

$$-1 \leq \sin(1/x) \leq 1.$$

Multiplying by $\ln(1+x)$ for $x > 0$, we get

$$-\ln(1+x) \leq \ln(1+x)\sin(1/x) \leq \ln(1+x).$$

Since

$$\lim_{x\to 0+} -\ln(1+x) = \lim_{x\to 0+} \ln(1+x) = \ln(1) = 0,$$

the squeeze theorem tells us that

$$\lim_{x\to 0^+} \ln(1+x)\sin(1/x) = 0.$$

(c)

$$\lim_{x\to 0}\left(\frac{1}{x^2+x} + \frac{1}{x^2-x}\right) = \lim_{x\to 0}\left(\frac{1}{x(x+1)} + \frac{1}{x(x-1)}\right) = \lim_{x\to 0}\left(\frac{(x-1)+(x+1)}{x(x+1)(x-1)}\right) = \lim_{x\to 0}\left(\frac{2x}{x(x+1)(x-1)}\right) = \lim_{x\to 0}\left(\frac{2}{(x+1)(x-1)}\right) = \frac{2}{(0+1)(0-1)} = -2.$$

(d)

$$\lim_{x\to 0} \frac{3^{2x}-1}{3^x-1} = \lim_{x\to 0} \frac{(3^x-1)(3^x+1)}{3^x-1} = \lim_{x\to 0}(3^x+1) = 3^0 + 1 = 2.$$

(5) (a) From the definition of the derivative,

$$f'(3) = \lim_{x\to 3} \frac{f(x)-f(3)}{x-3} = \lim_{x\to 3} \frac{\frac{x-1}{x+1} - \frac{2}{4}}{x-3} = \lim_{x\to 3} \frac{\frac{x-1}{x+1} - \frac{1}{2}}{x-3} = \lim_{x\to 3} \frac{\frac{2x-2-x-1}{2(x+1)}}{x-3} = \lim_{x\to 3} \frac{x-3}{2(x+1)(x-3)} = \lim_{x\to 3} \frac{1}{2(x+1)} = \frac{1}{8}.$$

(b) Since $f'(3) > 0$, the tangent point upwards and hence $f(x)$ is increasing near $x = 3$.