2. (a) $\lim_{x\to a}[f(x)p(x)]$ is an indeterminate form of type $0\cdot\infty$.

(b) When x is near a, $p(x)$ is large and $h(x)$ is near 1, so $h(x)p(x)$ is large. Thus, $\lim_{x\to a}[h(x)p(x)]=\infty$.

(c) When x is near a, $p(x)$ and $q(x)$ are both large, so $p(x)q(x)$ is large. Thus, $\lim_{x\to a}[p(x)q(x)]=\infty$.

4. (a) $\lim_{x\to a}[f(x)]^{g(x)}$ is an indeterminate form of type 0^0.

(b) If $y=[f(x)]^{p(x)}$, then $\ln y=p(x)\ln f(x)$. When x is near a, $p(x)\to\infty$ and $\ln f(x)\to-\infty$, so $\ln y\to-\infty$.
Therefore, $\lim_{x\to a}[f(x)]^{p(x)}=\lim_{x\to a}y=\lim_{x\to a}e^{\ln y}=0$, provided f^p is defined.

(c) $\lim_{x\to a}[h(x)]^{p(x)}$ is an indeterminate form of type 1^∞.

(d) $\lim_{x\to a}[p(x)]^{f(x)}$ is an indeterminate form of type ∞^0.

(e) If $y=[p(x)]^{q(x)}$, then $\ln y=q(x)\ln p(x)$. When x is near a, $q(x)\to\infty$ and $\ln p(x)\to\infty$, so $\ln y\to\infty$. Therefore,
$\lim_{x\to a}[p(x)]^{q(x)}=\lim_{x\to a}y=\lim_{x\to a}e^{\ln y}=\infty$.

(f) $\lim_{x\to a}\sqrt[q(x)]{p(x)}=\lim_{x\to a}[p(x)]^{1/q(x)}$ is an indeterminate form of type ∞^0.

6. From the graphs of f and g, we see that $\lim_{x\to 2}f(x)=0$ and $\lim_{x\to 2}g(x)=0$, so l'Hospital's Rule applies.

$$\lim_{x\to 2}\frac{f(x)}{g(x)}=\lim_{x\to 2}\frac{f'(x)}{g'(x)}=\frac{\lim_{x\to 2}f'(x)}{\lim_{x\to 2}g'(x)}=\frac{f'(2)}{g'(2)}=\frac{1.5}{-1}=-\frac{3}{2}$$

12. This limit has the form $\frac{0}{0}$. $\lim_{x\to 0}\frac{\sin 4x}{\tan 5x}\overset{H}{=}\lim_{x\to 0}\frac{4\cos 4x}{5\sec^2(5x)}=\frac{4(1)}{5(1)^2}=\frac{4}{5}$

28. This limit has the form $\frac{0}{0}$.

$$\lim_{x\to 0}\frac{x-\sin x}{x-\tan x}\overset{H}{=}\lim_{x\to 0}\frac{1-\cos x}{1-\sec^2 x}\overset{H}{=}\lim_{x\to 0}\frac{-(-\sin x)}{-2\sec x\,(\sec x\,\tan x)}=-\frac{1}{2}\lim_{x\to 0}\frac{\sin x\left(\dfrac{\cos x}{\sin x}\right)}{\sec^2 x}$$
$$=-\tfrac{1}{2}\lim_{x\to 0}\cos^3 x=-\tfrac{1}{2}(1)^3=-\tfrac{1}{2}$$

Another method is to write the limit as $\lim_{x\to 0}\dfrac{1-\dfrac{\sin x}{x}}{1-\dfrac{\tan x}{x}}$.

41. This limit has the form $\infty\cdot 0$. We'll change it to the form $\frac{0}{0}$.

$$\lim_{x\to\infty}x\sin(\pi/x)=\lim_{x\to\infty}\frac{\sin(\pi/x)}{1/x}\overset{H}{=}\lim_{x\to\infty}\frac{\cos(\pi/x)(-\pi/x^2)}{-1/x^2}=\pi\lim_{x\to\infty}\cos(\pi/x)=\pi(1)=\pi$$

55. $y = x^{\sqrt{x}} \quad \Rightarrow \quad \ln y = \sqrt{x}\,\ln x$, so

$$\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} \sqrt{x}\,\ln x = \lim_{x\to 0^+} \frac{\ln x}{x^{-1/2}} \overset{\text{H}}{=} \lim_{x\to 0^+} \frac{1/x}{-\frac{1}{2}x^{-3/2}} = -2 \lim_{x\to 0^+} \sqrt{x} = 0 \quad \Rightarrow$$

$$\lim_{x\to 0^+} x^{\sqrt{x}} = \lim_{x\to 0^+} e^{\ln y} = e^0 = 1.$$

61. $y = x^{1/x} \quad \Rightarrow \quad \ln y = (1/x)\,\ln x \quad \Rightarrow \quad \lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln x}{x} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{1/x}{1} = 0 \quad \Rightarrow$

$$\lim_{x\to\infty} x^{1/x} = \lim_{x\to\infty} e^{\ln y} = e^0 = 1$$

72. This limit has the form $\frac{\infty}{\infty}$. $\lim_{x\to\infty} \frac{\ln x}{x^p} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{1/x}{px^{p-1}} = \lim_{x\to\infty} \frac{1}{px^p} = 0$ since $p > 0$.

73. $\lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{1}{\frac{1}{2}(x^2+1)^{-1/2}(2x)} = \lim_{x\to\infty} \frac{\sqrt{x^2+1}}{x}$. Repeated applications of l'Hospital's Rule result in the original limit or the limit of the reciprocal of the function. Another method is to try dividing the numerator and denominator by x: $\lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}} = \lim_{x\to\infty} \frac{x/x}{\sqrt{x^2/x^2+1/x^2}} = \lim_{x\to\infty} \frac{1}{\sqrt{1+1/x^2}} = \frac{1}{1} = 1$

4. Call the two numbers x and y. Then $x + y = 16$, so $y = 16 - x$. Call the sum of their squares S. Then $S = x^2 + y^2 = x^2 + (16-x)^2 \quad \Rightarrow \quad S' = 2x + 2(16-x)(-1) = 2x - 32 + 2x = 4x - 32$. $S' = 0 \quad \Rightarrow \quad x = 8$. Since $S'(x) < 0$ for $0 < x < 8$ and $S'(x) > 0$ for $x > 8$, there is an absolute minimum at $x = 8$. Thus, $y = 16 - 8 = 8$ and $S = 8^2 + 8^2 = 128$.

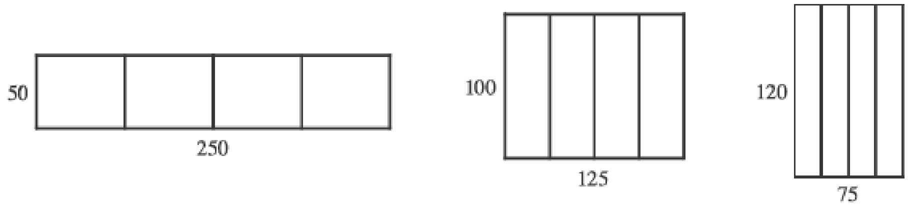
8. If the rectangle has dimensions x and y, then its area is $xy = 1000\ \text{m}^2$, so $y = 1000/x$. The perimeter $P = 2x + 2y = 2x + 2000/x$. We wish to minimize the function $P(x) = 2x + 2000/x$ for $x > 0$. $P'(x) = 2 - 2000/x^2 = (2/x^2)(x^2 - 1000)$, so the only critical number in the domain of P is $x = \sqrt{1000}$. $P''(x) = 4000/x^3 > 0$, so P is concave upward throughout its domain and $P(\sqrt{1000}) = 4\sqrt{1000}$ is an absolute minimum value. The dimensions of the rectangle with minimal perimeter are $x = y = \sqrt{1000} = 10\sqrt{10}$ m. (The rectangle is a square.)

10. We need to maximize P for $I \geq 0$. $\quad P(I) = \frac{100I}{I^2+I+4} \quad \Rightarrow$

$$P'(I) = \frac{(I^2+I+4)(100) - 100I(2I+1)}{(I^2+I+4)^2} = \frac{100(I^2+I+4-2I^2-I)}{(I^2+I+4)^2} = \frac{-100(I^2-4)}{(I^2+I+4)^2} = \frac{-100(I+2)(I-2)}{(I^2+I+4)^2}.$$

$P'(I) > 0$ for $0 < I < 2$ and $P'(I) < 0$ for $I > 2$. Thus, P has an absolute maximum of $P(2) = 20$ at $I = 2$.

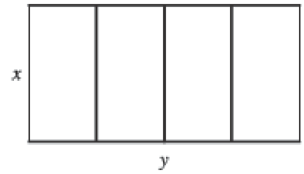
11. (a)



The areas of the three figures are 12,500, 12,500, and 9000 ft^2. There appears to be a maximum area of at least 12,500 ft^2.

(b) Let x denote the length of each of two sides and three dividers.

Let y denote the length of the other two sides.



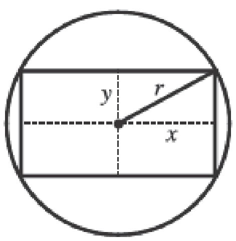
(c) Area $A = \text{length} \times \text{width} = y \cdot x$

(d) Length of fencing $= 750 \quad \Rightarrow \quad 5x + 2y = 750$

(e) $5x + 2y = 750 \quad \Rightarrow \quad y = 375 - \frac{5}{2}x \quad \Rightarrow \quad A(x) = \left(375 - \frac{5}{2}x\right)x = 375x - \frac{5}{2}x^2$

(f) $A'(x) = 375 - 5x = 0 \quad \Rightarrow \quad x = 75$. Since $A''(x) = -5 < 0$ there is an absolute maximum when $x = 75$. Then $y = \frac{375}{2} = 187.5$. The largest area is $75\left(\frac{375}{2}\right) = 14{,}062.5 \text{ ft}^2$. These values of x and y are between the values in the first and second figures in part (a). Our original estimate was low.

23.



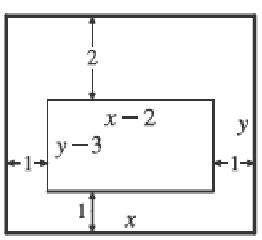
The area of the rectangle is $(2x)(2y) = 4xy$. Also $r^2 = x^2 + y^2$ so $y = \sqrt{r^2 - x^2}$, so the area is $A(x) = 4x\sqrt{r^2 - x^2}$. Now

$$A'(x) = 4\left(\sqrt{r^2 - x^2} - \frac{x^2}{\sqrt{r^2 - x^2}}\right) = 4\,\frac{r^2 - 2x^2}{\sqrt{r^2 - x^2}}.$$

The critical number is $x = \frac{1}{\sqrt{2}}r$. Clearly this gives a maximum.

$y = \sqrt{r^2 - \left(\frac{1}{\sqrt{2}}r\right)^2} = \sqrt{\frac{1}{2}r^2} = \frac{1}{\sqrt{2}}r = x$, which tells us that the rectangle is a square. The dimensions are $2x = \sqrt{2}\,r$ and $2y = \sqrt{2}\,r$.

34.



$xy = 180$, so $y = 180/x$. The printed area is

$(x - 2)(y - 3) = (x - 2)(180/x - 3) = 186 - 3x - 360/x = A(x)$.

$A'(x) = -3 + 360/x^2 = 0$ when $x^2 = 120 \quad \Rightarrow \quad x = 2\sqrt{30}$. This gives an absolute maximum since $A'(x) > 0$ for $0 < x < 2\sqrt{30}$ and $A'(x) < 0$ for $x > 2\sqrt{30}$. When $x = 2\sqrt{30}$, $y = 180/(2\sqrt{30})$, so the dimensions are $2\sqrt{30}$ in. and $90/\sqrt{30}$ in.

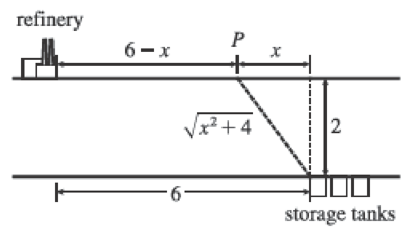
40. The volume and surface area of a cone with radius r and height h are given by $V = \frac{1}{3}\pi r^2 h$ and $S = \pi r\sqrt{r^2+h^2}$.

We'll minimize $A = S^2$ subject to $V = 27$. $V = 27 \Rightarrow \frac{1}{3}\pi r^2 h = 27 \Rightarrow r^2 = \dfrac{81}{\pi h}$ **(1)**.

$A = \pi^2 r^2(r^2+h^2) = \pi^2\left(\dfrac{81}{\pi h}\right)\left(\dfrac{81}{\pi h}+h^2\right) = \dfrac{81^2}{h^2} + 81\pi h$, so $A' = 0 \Rightarrow \dfrac{-2\cdot 81^2}{h^3} + 81\pi = 0 \Rightarrow$

$81\pi = \dfrac{2\cdot 81^2}{h^3} \Rightarrow h^3 = \dfrac{162}{\pi} \Rightarrow h = \sqrt[3]{\dfrac{162}{\pi}} = 3\sqrt[3]{\dfrac{6}{\pi}} \approx 3.722$. From **(1)**, $r^2 = \dfrac{81}{\pi h} = \dfrac{81}{\pi\cdot 3\sqrt[3]{6/\pi}} = \dfrac{27}{\sqrt[3]{6\pi^2}} \Rightarrow$

$r = \dfrac{3\sqrt{3}}{\sqrt[6]{6\pi^2}} \approx 2.632$. $A'' = 6\cdot 81^2/h^4 > 0$, so A and hence S has an absolute minimum at these values of r and h.



49. There are $(6-x)$ km over land and $\sqrt{x^2+4}$ km under the river.

We need to minimize the cost C (measured in \$100,000) of the pipeline.

$C(x) = (6-x)(4) + \left(\sqrt{x^2+4}\right)(8) \Rightarrow$

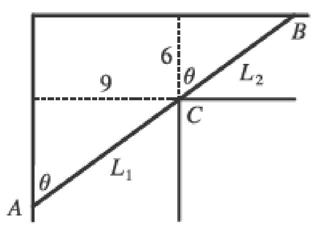
$C'(x) = -4 + 8\cdot\frac{1}{2}(x^2+4)^{-1/2}(2x) = -4 + \dfrac{8x}{\sqrt{x^2+4}}$.

$C'(x) = 0 \Rightarrow 4 = \dfrac{8x}{\sqrt{x^2+4}} \Rightarrow \sqrt{x^2+4} = 2x \Rightarrow x^2+4 = 4x^2 \Rightarrow 4 = 3x^2 \Rightarrow x^2 = \frac{4}{3} \Rightarrow$

$x = 2/\sqrt{3}$ $[0 \le x \le 6]$. Compare the costs for $x = 0, 2/\sqrt{3}$, and 6. $C(0) = 24 + 16 = 40$,

$C\left(2/\sqrt{3}\right) = 24 - 8/\sqrt{3} + 32/\sqrt{3} = 24 + 24/\sqrt{3} \approx 37.9$, and $C(6) = 0 + 8\sqrt{40} \approx 50.6$. So the minimum cost is about \$3.79 million when P is $6 - 2/\sqrt{3} \approx 4.85$ km east of the refinery.

70.



Paradoxically, we solve this maximum problem by solving a minimum problem. Let L be the length of the line ACB going from wall to wall touching the inner corner C. As $\theta \to 0$ or $\theta \to \frac{\pi}{2}$, we have $L \to \infty$ and there will be an angle that makes L a minimum. A pipe of this length will just fit around the corner.

From the diagram, $L = L_1 + L_2 = 9\csc\theta + 6\sec\theta \Rightarrow dL/d\theta = -9\csc\theta\cot\theta + 6\sec\theta\tan\theta = 0$ when

$6\sec\theta\tan\theta = 9\csc\theta\cot\theta \Leftrightarrow \tan^3\theta = \frac{9}{6} = 1.5 \Leftrightarrow \tan\theta = \sqrt[3]{1.5}$. Then $\sec^2\theta = 1 + \left(\frac{3}{2}\right)^{2/3}$ and

$\csc^2\theta = 1 + \left(\frac{3}{2}\right)^{-2/3}$, so the longest pipe has length $L = 9\left[1+\left(\frac{3}{2}\right)^{-2/3}\right]^{1/2} + 6\left[1+\left(\frac{3}{2}\right)^{2/3}\right]^{1/2} \approx 21.07$ ft.

Or, use $\theta = \tan^{-1}\left(\sqrt[3]{1.5}\right) \approx 0.853 \Rightarrow L = 9\csc\theta + 6\sec\theta \approx 21.07$ ft.