Calculus I: Final Exam

May 3, 2015

Name: Solutions

- Write your solutions in the space provided. Continue on the back for more space.
- The last few pages are left blank for scratch work. You can detach them.
- You must show your work. Just writing the final answer will receive little credit.
- The exam contains 10 problems.
- **Good luck!**

Question:	1	2	3	4	5	6	7	8	9	10	Total
Points:	10	10	10	10	10	10	10	10	10	10	100
Score:											

1. (10 points) Differentiate the following functions.

(a) $\sin(x)e^x$

$$\cos(x)e^x + \sin(x)e^x$$

(b) x^x

$$y = x^x$$

$$\ln y = x \ln x$$

$$\frac{y'}{y} = \ln x + 1$$

$$y' = x^x(\ln x + 1)$$

2. (10 points) The following table lists the values of $f(x)$ and $f'(x)$ for a differentiable function f.

x	0	1	2	3	4
$f(x)$	0	4	5	2	-3
$f'(x)$	3	2	0	-1	-5

(a) Let $g(x) = f(x)^3$. Find $g'(2)$.

$$g'(x) = 3f(x)^2 \cdot f'(x)$$

$$g'(2) = 3 \times f(2)^2 \cdot f'(2)$$

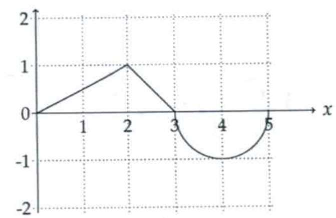
$$= 3 \times 25 \times 0 = 0$$

(b) Let $h(x) = \frac{f(x)}{x}$. Find $h'(1)$.

$$h'(x) = \frac{f'(x) \cdot 1 - 1 \cdot f(x)}{x^2}$$

$$h'(1) = \frac{2-0}{1} = 2$$

3. (10 points) The following graph depicts a function f defined on $[0,5]$. From 0 to 2 and 2 to 3, the graph is linear, and from 3 to 5, it is semi-circular.



(a) Find $\int_0^5 f(x)dx$.

$$\frac{1}{2} \times 3 - \frac{\pi}{2}$$

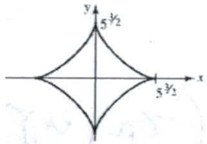
(b) Find $\int_0^1 f(3x+2)dx$.

$$u = 3x+2$$

$$\int_2^5 \frac{f(u)}{3} du = \frac{1}{3}\left(\frac{1}{2} - \frac{\pi}{2}\right)$$

4. (10 points) The *astroid*, shown in the picture here, is defined by the equation

$$x^{2/3} + y^{2/3} = 5.$$



Find the equation of the tangent line to the astroid at the point ~~(4,1)~~. $(8,1)$.

$$\frac{2}{3} x^{-1/3} + \frac{2}{3} y^{-1/3} \cdot \frac{dy}{dx} = 0$$

$$\frac{1}{2} + \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{1}{2}$$

$2y - x = -5$ ← tangent line

$2y - x + 5 = 0$

5. (10 points) Evaluate the following.

(a) $\lim_{x \to +\infty} \frac{3x^2 - 4x}{x^2 + 1}$

Ans = 3

(b) $\lim_{x \to 0+} (1 + 3x)^{1/x}$

(take ln & use L'Hôpital)

Ans = e^3

6. (10 points) Find the derivative of $\ln(\cos(x))$. Use your result to find the area under the curve $y = \tan(x)$ from $x = 0$ to $x = \pi/4$.

$f(x) = \ln(\cos x)$

$f'(x) = -\frac{\sin(x)}{\cos(x)} = -\tan x$

$\Rightarrow \int \tan x \, dx = -\ln(\cos x).$

$\int_0^{\pi/4} \tan x \, dx = -\ln(\cos x)\Big|_0^{\pi/4}$

$= \ln(\sqrt{2})$

7. (10 points) Let $f(x) = x^2(2 - x^2)$. $= 2x^2 - x^4$

(a) Find all the critical points of $f(x)$.

Ans: $f'(x) = 4x - 4x^3$

$f'(x) = 0 \Rightarrow x = 0, 1, -1.$

(b) Pick one of the critical points. Determine if $f(x)$ is concave up or concave down at that point.

$f''(x) = 4 - 12x^2$

$f''(0) = 4 \Rightarrow$ conc. up

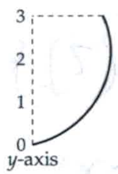
$f''(1) = -8 \Rightarrow$ conc down

$f''(-1) = -8 \rightarrow$ conc. down.

8. (10 points) The shape of a bowl is obtained by rotating the curve

$$x^2 + (y-2)^2 = 4$$

about the y-axis, where we take $0 \le y \le 3$ and $x \ge 0$. The figure to the right shows the curve.



How much liquid does the bowl hold when it is filled completely up to the brim?

Skip.

9. (10 points) Oil is leaking from a ship in the ocean and spreading evenly in a thin, expanding disk. Suppose at a given instant, the radius of the spill is 10 meters and the rate of change of the area of the spill is 60π square meters per minute. Find the rate of change of the radius at that instant and use it to estimate the radius of the spill 10 seconds later.

Ans : $\frac{dr}{dt} = 3 \frac{\text{meters}}{\text{min}}$

After 10 sec,

$r \approx 10.5$ m.

10. (10 points) Suppose the sum of two non-negative numbers is 4. What is the maximum and minimum possible value of the sum of their cubes?

max : $0^3 + 4^3 = 64$

min : $2^3 + 2^3 = 16$

($x + y = 4$ $\quad x \geq 0$, $y \geq 0$

max/min $x^3 + y^3$

$f(x) = x^3 + (10-x)^3 \quad 0 \leq x \leq 4$

Diff. set $= 0$

Get crit pts. $(x=2)$

compare $f(0)$, $f(2)$, $f(4)$.

Scratch Work