Analysis and optimization: Midterm 2

Spring 2016

- Answer the questions in the space provided.
- Give concise but adequate reasoning unless asked otherwise.
- You may use any statement from class, textbook, or homework without proof, but you must clearly write the statements you use.
- The exam contains 6 questions.

Name: Solutions.

Section: ☐ 8:40–9:55 ☐ 10:10–11:25

Question	Points	Score
1	5	
2	9	
3	8	
4	8	
5	10	
6	10	
Total:	50	

1. (a) (2 points) State the definition of a convex function.

$f: S \to \mathbb{R}$ is convex if S is a convex set and for every $\bar{x}, \bar{y}$ in S and λ in $[0,1]$, we have

$$\lambda f(\bar{x}) + (1-\lambda) f(\bar{y}) \geq f(\lambda \bar{x} + (1-\lambda)\bar{y}).$$

(b) (3 points) Suppose $f: \mathbf{R}^n \to \mathbf{R}$ is a convex function and a is a real number. Show that the set $\{\vec{x} \in \mathbf{R}^n \mid f(\vec{x}) \leq a\}$ is convex. Call the set K.

Let $\bar{x}, \bar{y} \in K$ and $\lambda \in [0,1]$.

We want to show that $\lambda\bar{x} + (1-\lambda)\bar{y}$ is in K.

$$\begin{aligned} f(\lambda\bar{x} + (1-\lambda)\bar{y}) &\leq \lambda f(\bar{x}) + (1-\lambda) f(\bar{y}) \\ &\leq \lambda a + (1-\lambda) a \\ &= a. \end{aligned}$$

So $\lambda\bar{x} + (1-\lambda)\bar{y}$ lies in K.

2. Give examples of the following:

(a) (3 points) A function with gradient $(1, -1)$ and Hessian $\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$ at (e, π).

$$(x-e) - (y-\pi) + \frac{1}{2}\left(2(x-e)^2 + 2(x-e)(y-\pi) + 3(y-\pi)^2\right)$$

(b) (3 points) A function on $\mathbf{R}^2$ with a critical point at $(0, 0)$ which is neither a local minimum nor a local maximum.

xy or x^2-y^2 or x^3+y^3

(c) (3 points) A convex function on $\mathbf{R}^2$ which is not strictly convex.

A constant function or a linear function are the easiest examples.

3. (8 points) Consider the equations

$$x^2 + y^2 = u, \quad x^3 + y^3 = v.$$

Show that we can express x and y as functions of u and v around the point $(x, y, u, v) = (1, 2, 5, 9)$ and find the partial derivatives $\frac{\partial x}{\partial u}$, $\frac{\partial x}{\partial v}$, $\frac{\partial y}{\partial u}$, and $\frac{\partial y}{\partial v}$ at this point.

Set $F_1 = x^2+y^2-u$ & $F_2 = x^3+y^3-v$.

Then
$$\frac{\partial F}{\partial(x,y)} = \begin{pmatrix} 2x & 2y \\ 3x^2 & 3y^2 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 4 \\ 3 & 12 \end{pmatrix} \quad \text{at} \quad (x,y) = (1,2).$$

Since this matrix is invertible, by the implicit function theorem, we can write x, y as functions of u, v around $(1,2,5,9)$.

We have the equation

$$\frac{\partial F}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(u,v)} = -\frac{\partial F}{\partial(u,v)}. \quad \text{so at } (1,2,5,9):$$

$$\begin{pmatrix} 2 & 4 \\ 3 & 12 \end{pmatrix} \begin{pmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{pmatrix} = -\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

so
$$\begin{pmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 3 & 12 \end{pmatrix}^{-1}$$

$$= \frac{1}{12}\begin{pmatrix} 12 & -4 \\ -3 & 2 \end{pmatrix}.$$

$$= 4\begin{pmatrix} 1 & -1/3 \\ -1/4 & -1/6 \end{pmatrix}$$

4. Let A be a symmetric $n \times n$ matrix and B any $m \times n$ matrix.

(a) (2 points) Show that $B^T AB$ is symmetric.

$$(B^T AB)^T = B^T A^T B^{TT}$$
$$= B^T AB \qquad \text{since } A^T = A.$$

So $B^T AB$ is symmetric.

(b) (4 points) Suppose A is positive definite. Show that $B^T AB$ is positive semi-definite.

A positive def $\Rightarrow$ $\bar{x}^T A \bar{x} \geqslant 0$ for every $\bar{x}$.

Now, $\bar{x}^T B^T A B \bar{x} = (B\bar{x})^T A (B\bar{x}).$

$= \bar{y}^T A \bar{y} \geqslant 0.$, where $\bar{y} = B\bar{x}$.

So, for any $\bar{x}$ we got $\bar{x}^T (B^T AB) \bar{x} \geq 0$

$\Rightarrow$ $B^T AB$ is positive semidefinite.

(c) (2 points) What condition on B will ensure that $B^T AB$ is positive definite?

In (b), note that we'd have strict >0 if

$B\bar{x} \neq 0$ for $\bar{x} \neq 0$. So $B^T AB$ will be pos. def

if $\underline{\text{rk } B = m}$.

(i.e. $B\bar{x} \neq 0$ for any $\bar{x} \neq 0$)

Equivalently, $\underline{\ker(B) = 0}$ or $\underline{\text{nullity}(B) = 0}$.

or B is <u>left</u> <u>invertible</u>.

5. (10 points) Consider the function

$$f(x,y,z) = x^3 + y^3 - 3xy + z^2 - 2z.$$

Find all the critical points of f and classify each one as a local maximum, local minimum, or saddle point.

$$\nabla f\,(x,y,z) = (3x^2-3y,\ 3y^2-3x,\ 2z-2)$$

$$= (0,0,0) \quad \text{means}$$

$2z = 2 \Rightarrow z = 1$

$\left.\begin{array}{l} x^2 = y \\ y^2 = x \end{array}\right\}$ $\quad x^4 = x \Rightarrow x = 0$ or $x = 1$

$\qquad\qquad\qquad\qquad \Downarrow \qquad\qquad \Downarrow$

$\qquad\qquad\qquad\qquad y=0 \qquad\quad y=1.$

So critical points are $(0,0,1)$ and $(1,1,1)$.

$$\mathrm{Hess}(f) = \begin{pmatrix} 6x & -3 & 0 \\ -3 & 6y & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

At $\underline{(0,0,1)}$: $\begin{pmatrix} 0 & -3 & 0 \\ -3 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ Leading principal minors: $0, -9, -18$

$\Rightarrow$ indefinite.

$\Rightarrow$ <u>Saddle point</u>.

At $\underline{(1,1,1)}$ $\begin{pmatrix} 6 & -3 & 0 \\ -3 & 6 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ Leading principal minors: $6, 27, 54$

$\Rightarrow$ pos. definite

$\Rightarrow$ <u>Local minimum</u>.

6. (10 points) The plane $8x-5y+z=5$ and the cylinder $x^2+y^2=1$ intersect in an ellipse. What are the maximum and minimum values of the function $f(x,y,z)=y+z$ on this ellipse and where are they attained?

Constraints : $8x-5y+z-5=0$ call $g_1(x,y,z)=0$

$x^2+y^2-1=0$ call $g_2(x,y,z)=0$

$\nabla g_1 = (8,-5,1)$

$\nabla g_2 = (2x, 2y, 0).$

$\nabla f = (0,1,1).$

Since the domain is <u>compact</u> and f is continuous, min/max exist!

At max/min there exist λ_1, λ_2 such that

$(0,1,1) = \lambda_1 (8,-5,1) + \lambda_2 (2x, 2y, 0).$

$$\left.\begin{aligned} 0 &= 8\lambda_1 + 2x\lambda_2 \\ 1 &= -5\lambda_1 + 2y\lambda_2 \end{aligned}\right\}$$

$1 = \lambda_1$

$2x\lambda_2 = -8$

$2y\lambda_2 = 5$

$\Downarrow$

also $x^2+y^2=1$ & $8x-5y+z=5$

$\frac{-8}{6} = \frac{x}{y} \Rightarrow y = -\frac{6x}{8} = -\frac{3}{4}x$

$x^2+y^2=1 \Rightarrow x^2 + \frac{36}{64}x^2 = 1 \Rightarrow x^2 = \frac{64}{100} = \frac{16}{25}$

so $x = \frac{4}{5}$ or $\frac{-4}{5}$

$x=\frac{4}{5} \Rightarrow y = \frac{-3}{5}, \quad z = \frac{-22}{5}$ & $f(x,y,z) = 15.$ $\Leftarrow$ MAX

$x=\frac{-4}{5} \Rightarrow y = 3/5, \quad z = \frac{72}{5}$ & $f(x,y,z) = -5.$ $\Leftarrow$ MIN.

So max at $(4/5, -3/5, -22/5)$

min at $(-4/5, 3/5, 72/5)$