Analysis and optimization: Midterm 1

Spring 2016

- Answer the questions in the space provided.
- Give concise but adequate reasoning unless asked otherwise.
- You may use any statement from class, textbook, or homework without proof, but you must clearly write the statements you use.
- The exam contains 6 questions.
- At the end, there are some blank pages for scratch work. You may detatch them.

Name: Solutions.

Section: ☐ 8:40–9:55 ☐ 10:10–11:25

Question	Points	Score
1	8	
2	10	
3	8	
4	6	
5	6	
6	12	
Total:	50	

1. Consider the matrix

$$M = \begin{pmatrix} 1 & 3 & 0 & 1 \\ 1 & 3 & -2 & -2 \\ 0 & 0 & 2 & 3 \end{pmatrix}.$$

(a) (3 points) Find the rank of M.

Do row reduction.

$$\begin{pmatrix} 1 & 3 & 0 & 1 \\ 1 & 3 & -2 & -2 \\ 0 & 0 & 2 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 3 & 0 & 1 \\ 0 & 0 & -2 & -3 \\ 0 & 0 & 2 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & 3/2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Since there are 2 non-zero rows in the row-reduced form, the rank is 2.

(b) (3 points) Find all $\vec{x}$ such that $M\vec{x} = \vec{0}$.

We may work with the row reduced form. Note that x_2 and x_4 are the free (non-pivot) variables. The equations are

$$x_1 + 3x_2 + x_4 = 0 \quad ; \quad x_3 + \frac{3}{2}x_4 = 0.$$

Solving for the non-free variables gives all the solutions:

$$\begin{pmatrix} -3x_2 - x_4 \\ x_2 \\ -3/2\, x_4 \\ x_4 \end{pmatrix} \quad \text{for any } x_2, x_4.$$

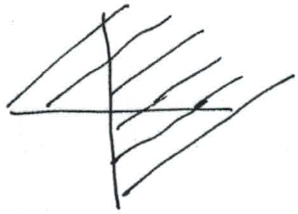
(c) (2 points) Are the rows of M linearly dependent or independent?

The rows are linearly dependent.

(Rank is 2 but there are 3 rows.)

2. No justification is necessary for this question.

(a) (2 points) Is the set $S = \{(x, y) \in \mathbf{R}^2 \mid x \geq 0 \text{ or } y \geq 0\}$ convex or not convex?



Not convex.

(b) (2 points) Is the set $S = \{(x, y) \in \mathbf{R}^2 \mid 1 < \sin(x^2 + y^2) < 3\}$ open or not open?

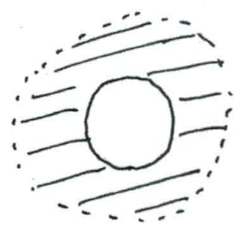
Open. In fact, it is empty!

(c) (2 points) Is the set $S = \left\{\frac{1}{n}, \text{ for } n = 1, 2, 3, \ldots\right\}$ compact or not compact?

Not compact. 0 is a boundary point, but $0 \notin S$.
So S is not closed.

(d) (2 points) Give an example of a subset of $\mathbf{R}^2$ which is neither open nor closed.

$\{(x,y) \in \mathbb{R}^2 \mid 1 \leq x^2+y^2 < 2\}$



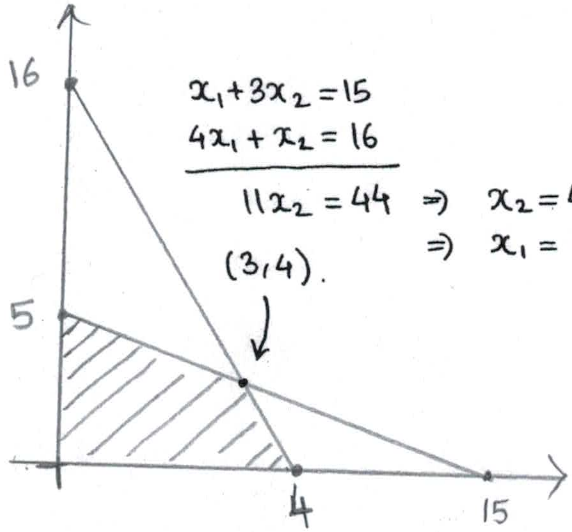
(e) (2 points) Give an example of a bounded S and a continuous function $f : S \to \mathbf{R}$ such that $f(S)$ is unbounded.

$S = (0,1]$, $f(x) = \frac{1}{x}$.

Then $f(S) = [1, +\infty)$

3. (a) (4 points) By plotting the feasible set on a graph, maximize $z = 3x_1 + 2x_2$ subject to

$$x_1 + 3x_2 \le 15, \quad 4x_1 + x_2 \le 16, \quad 0 \le x_1, \quad 0 \le x_2.$$



$$\begin{array}{r} x_1 + 3x_2 = 15 \\ 4x_1 + x_2 = 16 \\ \hline 11x_2 = 44 \end{array} \Rightarrow x_2 = 4 \Rightarrow x_1 = 3$$

Vertex	z
(0,0)	0
(4,0)	~~33~~ 12
(0,5)	10
(3,4)	17

So max. z is 17 at $x_1 = 3$, $x_2 = 4$.

(b) (2 points) Formulate (but do not solve) the dual problem.

Minimize $w = 15y_1 + 16y_2$ subject to

$$y_1 + 4y_2 \ge 3 \qquad y_1 \ge 0$$
$$3y_1 + y_2 \ge 2 \qquad y_2 \ge 0$$

(c) (2 points) Using that (5/11, 7/11) is the solution to the dual problem, answer the following. What would the optimal value of z in the original problem be if the second constraint was changed to $4x_1 + x_2 \le 16 + \epsilon$?

Optimal $y_2 = 7/11$ $\Rightarrow$ shadow price of ~~y_2~~ the second constraint is $\frac{7}{11}$.

$\Rightarrow$ An increase of ε leads to an increase of $\frac{7}{11}\varepsilon$ in the optimal z. So new optimal

$$z = 17 + \frac{7}{11}\varepsilon$$

4. (a) (2 points) State the maximum theorem (also called Weierstrass's theorem).

Let S be compact and $f: S \to \mathbb{R}$ a continuous function.

Then f attains a maximum and a minimum value on S.

That is, there exist x_1, x_2 in S such that

$f(x_1) \geq f(y)$ for all y in S and $f(x_2) \leq f(y)$ for all y in S.

(b) (4 points) Find the maximum and minimum values of $f(x) = e^x + 4e^{-x}$ on $[0, 1]$.

Let us find the critical points.

$$f'(x) = e^x - 4e^{-x} = 0$$

$$\Rightarrow \quad e^x = \frac{4}{e^x}$$

$$\Rightarrow \quad e^{2x} = 4 \quad \Rightarrow \quad e^x = 2. \quad \Rightarrow \quad x = \ln 2.$$

Max / min are attained at end pts or critical pts.

So we test $0, \ln(2), 1$.

$f(0) = 5$

$f(\ln 2) = 2 + \frac{4}{2} = 4$

$f(1) = e + \frac{4}{e}.$

Sign of f':	$-$	$+$
Dir. of f:	$\searrow$	$\nearrow$

$0 \qquad \ln 2 \qquad 1$

So $\underline{\ln 2}$ is the $\underline{\text{min}}$.

To compare $f(0)$ & $f(1)$, let us use $2 < e < 3$

So ~~e~~ ~~$\frac{4}{e}$~~ $\frac{4}{2} > \frac{4}{e} > \frac{4}{3}$

So $e + \frac{4}{e} < 3 + 2 = 5.$ $\Rightarrow$ $\underline{0}$ is the $\underline{\text{max}}$.

(You could also estimate $e + \frac{4}{e}$ by taking a reasonable approx. value of e.)

5. Suppose $S \subset \mathbb{R}^n$ is a convex set and A is an $m \times n$ matrix. Let

$$T = \{\vec{y} \in \mathbb{R}^m \mid \vec{y} = A\vec{x} \text{ for some } \vec{x} \in S\}.$$

(a) (4 points) Using the definition of a convex set, show that T is convex.

To show that T is convex, we must show that for every $\bar{y}_1, \bar{y}_2 \in T$ and $\lambda \in [0,1]$, we have $\lambda \bar{y}_1 + (1-\lambda)\bar{y}_2 \in T$.

Let $\bar{y}_1 = A\bar{x}_1$ and $\bar{y}_2 = A\bar{x}_2$, where $\bar{x}_1, \bar{x}_2 \in S$.

Then $\lambda \bar{y}_1 + (1-\lambda)\bar{y}_2 = \lambda A\bar{x}_1 + (1-\lambda) A\bar{x}_2$
$= A(\lambda \bar{x}_1 + (1-\lambda)\bar{x}_2)$.

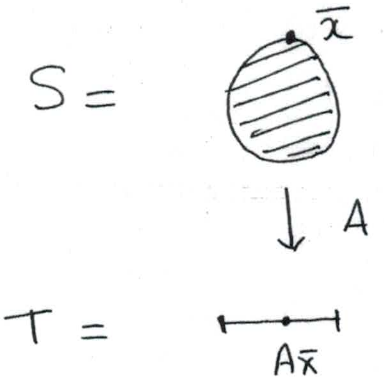
Since $\cancel{\bar{x}}$ S is convex, we know that $\lambda \bar{x}_1 + (1-\lambda)\bar{x}_2 \in S$.

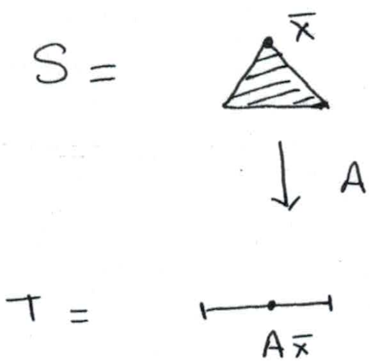
so $\lambda \bar{y}_1 + (1-\lambda)\bar{y}_2 \in T$.

(b) (2 points) Suppose $\vec{x}$ is an extreme point of S. Must $A\vec{x}$ be an extreme point of T? Justify your answer with a proof (if your answer is "Yes") or with a counter-example (if your answer is "No").

No. Here are two counter examples. In both examples $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ so

$A \begin{bmatrix} x \\ y \end{bmatrix} = x$ (A represents projection onto the x-axis).





6. An accounting firm has 900 hours of staff time and 100 hours of reviewing time available each week. The firm charges \$2000 for an audit and \$300 for a tax return. Each audit requires 100 hours of staff time and 10 hours of reviewing time. Each tax return requires 10 hours of staff time and 2 hours of reviewing time.

(a) (3 points) Formulate the linear programming problem to find the number of audits and tax returns to do each week to get the maximum revenue.

Let x be the number of audits and y the number of tax rets

Maximize $\quad Z = 2000x + 300y \quad$ subj to

$$100x + 10y \le 900$$
$$10x + 2y \le 100$$
$$0 \le x \quad 0 \le y.$$

(b) (5 points) Use the simplex method to solve the problem.

Let us use s and t to denote the slack variables.

Initial tableau:

	x	y	s	t	b	Basic	
	100	10	1	0	900	s	← departing.
	10	2	0	1	100	t.	
Z	−2000	−300	0	0	0		
	↑ entering						

Step 1 :

	x	y	s	t	b	Basic	
	1	$1/10$	$1/100$	0	9	x	
	0	1	$-1/10$	1	10	t.	← departing
Z	0	−100	20	0	18000		
		↑ entering					

Step 2 :

	x	y	s	t	b	Basic
	1	0	$1/50$	$-1/10$	8	x
	0	1	$-1/10$	1	10	y.
Z	0	0	10	100	19000	

DONE!

So the optimal solution is

$$x = 8$$
$$y = 10$$

for a revenue of \$19000

(c) (2 points) The firm has a new intern. Should they be directed to help the staff or the reviewers? Justify your answer.

The marginal utility of staff is \$10 whereas the marginal utility of reviewers is \$100.

i.e. an ε increase in reviewing time $\Rightarrow$ 100ε inc. in revenue.

So the intern should help the reviewers.

(d) (2 points) How much money (per hour) should the firm be willing to pay for a reviewer?

The firm should pay at most the shadow price for reviewing, i.e. \$100.