ANALYSIS AND OPTIMIZATION: MIDTERM 1 PRACTICE PROBLEMS (SOLUTIONS)

SPRING 2016

(1) Find the global minimum and maximum of the function $f(x) = x^3/3 + x^2/2 - 2x$ on the set $[-3, 3]$.

Solution. Differentiate, find the critical points, and evaluate $f(x)$ at the critical points to locate the min and max. $\square$

(2) Calculate the rank of the matrix

$$\begin{pmatrix} 2 & 1 & 3 & 7 \\ -1 & 4 & 3 & 1 \\ 3 & 2 & 5 & 11. \end{pmatrix}$$

Are columns 1, 2, and 3 linearly independent?

Solution. Use Gaussian elimination (row reduction). Columns 1, 2, and 3 are linearly dependent if and only if they are linearly dependent after row reduction, which means that none of them should become zero. In this case, they are linearly dependent. $\square$

(3) Give an example of the following or explain why an example does not exist.

(a) A continuous function $f : [0, 1] \to \mathbf{R}$ whose image is unbounded.
Solution. Since $[0, 1]$ is compact, f must have a maximum and a minimum (by Weierstrass's theorem). This means that the image must be bounded. So no such example exists. $\square$

(b) A continuous function $f : \mathbf{R} \to \mathbf{R}$ whose image is not closed.
Solution. Take $f(x) = e^x$. Then the image of f is $\mathbf{R}_{>0}$, which is not closed. $\square$

(c) A convex set without any extreme points.
Solution. Take the set to be $\mathbf{R}^n$. $\square$

(d) A convex set with infinitely many extreme points.
Solution. Take the circular disk $\{(x, y) \in \mathbf{R}^2 \mid x^2 + y^2 \leq 1\}$. $\square$

(e) A subset of $\mathbf{R}^2$ that is neither open nor closed.
Solution. Take the open circular disk $\{(x, y) \in \mathbf{R}^2 \mid x^2 + y^2 < 1\}$ union the upper half-circle $\{(x, y) \in \mathbf{R}^2 \mid x^2 + y^2 = 1, y \geq 0\}$. $\square$

(f) A subset of $\mathbf{R}$ that is both open and closed.
Solution. The empty set or $\mathbf{R}$. $\square$

(4) Suppose $S \subset \mathbf{R}^n$ is a convex set. Let A be an $m \times n$ matrix. Show that the set

$$T = \{\vec{y} \in \mathbf{R}^n \mid \vec{y} = A\vec{x} \text{ for some } \vec{x} \in S\}$$

is a convex set.

Solution. We must show that for any two points $\vec{y}_1$ and $\vec{y}_2$ in T, and any constant $\lambda \in [0, 1]$, the point $\lambda\vec{y}_1 + (1 - \lambda)\vec{y}_2$ lies in T. Since $\vec{y}_1 \in T$, there exists $\vec{x}_1 \in S$ such that $\vec{y}_1 = A\vec{x}_2$, and similarly, there exists $\vec{x}_2 \in S$ such that $\vec{y}_2 = A\vec{x}_2$. Now, we have

$$\lambda\vec{y}_1 + (1-\lambda)\vec{y}_2 = \lambda A\vec{x}_1 + (1-\lambda)A\vec{x}_2 = A(\lambda\vec{x}_1 + (1-\lambda)\vec{x}_2).$$

Since S is convex and $\vec{x}_1, \vec{x}_2 \in S$, we know that $\lambda\vec{x}_1 + (1-\lambda)\vec{x}_2 \in S$. Therefore, $\lambda\vec{y}_1 + (1-\lambda)\vec{y}_2 = A(\lambda\vec{x}_1 + (1-\lambda)\vec{x}_2)$ lies in T. □

(5) Use Gaussian elimination to solve the equations

$$\begin{aligned} x + y + 2z &= 1 \\ -x + y + 3 &= 0 \\ y + 3z &= 2. \end{aligned}$$

Solution. Left to you. □

(6) Find the determinant of

$$\begin{pmatrix} 2 & 3 & 1 & 0 \\ 4 & -2 & 0 & -3 \\ 8 & -1 & 2 & 1 \\ 1 & 0 & 3 & 4 \end{pmatrix}$$

Solution. Left to you. □

(7) Let $S \subset \mathbf{R}^n$ be convex, and $\vec{x}, \vec{y}, \vec{z} \in S$. Let $a, b, c \in \mathbf{R}$ be such that $0 \le a, b, c \le 1$ and $a + b + c = 1$. Show that $a\vec{x} + b\vec{y} + c\vec{z} \in S$.

Solution. Let $\lambda = a/(a+b)$. Note that $0 \le \lambda \le 1$ and $1 - \lambda = (b)/(a+b)$. Since $\vec{x}$ and $\vec{y}$ are in S, we get that

$$\vec{w} = \lambda\vec{x} + \lambda\vec{y} = (a\vec{x} + b\vec{y})/(a+b)$$

also lies in S.

Now take $\mu = (a+b)$. Then $0 \le \mu \le 1$ and $1 - \mu = c$. Since $\vec{w}$ and $\vec{z}$ are in S, we get that

$$(a+b)\vec{w} + c\vec{z} = a\vec{x} + b\vec{y} + c\vec{z}$$

also lies in S.

By repeating the same argument, we can prove that if $\vec{x}_1, \dots, \vec{x}_k \in S$ and $0 \le a_1, \dots, a_k \le 1$ are such that $a_1 + \cdots + a_k = 1$ then the vector $a_1\vec{x}_1 + \cdots + a_k\vec{x}_k$ also lies in S. □

(8) Do the word problem from LEF 9.3, problem 25. How much extra profit is obtained by increasing the allowed assembly time by an hour? Increasing painting time by an hour? Increasing packaging time by an hour?

Solution. Let x_1, x_2, x_3 be the number of bikes of type A, B, C. We get the following linear program. Maximize $45x_1 + 50x_2 + 55x_3$ subject to

$$\begin{aligned} 2x_1 + 2.5x_2 + 3x_3 &\le 4006 \\ 1.5x_1 + 2x_2 + x_3 &\le 2495 \\ x_1 + 0.75x_2 + 1.25x_3 &\le 1500 \\ &, 0 \le x_1, x_2, x_3. \end{aligned}$$

Using the simplex method, we get the following final tableau

x_1	x_2	x_3	s_1	s_2	s_3	b	Basic
0	1	0	2/3	8/21	-40/21	764	x_2
1	0	0	-4/3	2/3	8/3	322	x_1
0	0	1	2/3	-16/21	-4/21	484	x_3
0	0	0	10	50/7	100/7	79310	

Therefore, 322 bikes of type A, 764 bikes of type B, and 484 bikes of types C should be produced to get the maximum profit of $79310. The shadow prices of assembly, painting, and packaging ares $10, $50/7, and $100/7, respectively, which denote the extra profit obtained by increasing their time by an hour.

The numbers in this problem got quite nasty. Since you won't have a calculator, the numbers will be friendlier in the exam. □

(9) Do the following maximization problems using the simplex method (the link to LEF is on the course webpage)
  (a) LEF 9.3, problem 14
  (b) LEF 9.3 problem 18

For each of these problems, what is the shadow price associated with each constraint?

Solution. I will just give the final answers (but you will have to show your work on the exam!).
  (a) $x_1 = 51/7; x_2 = 18/7$ for the optimal $z = 87/7$ with shadow prices 5/7 and 1/7.
  (b) $x_1 = 30; x_2 = 24; x_3 = 5$ for the optimal $z = 99$ with shadow prices 12/13, 7/13, and 1/13.

□

(10) Minimize $z = 14x + 20y$ subject to the constraints $x + 2y \geq 4$, $7x + 6y \geq 20$, and $x, y \geq 0$ by plotting the feasible set.
Formulate and solve the dual problem graphically.
If we change the constraint (in the primal) to $x + 2y \geq 4 + \epsilon$, what would be the change in the optimal z?

Solution. For the first part, the optimal solution is $x = 2; y = 1$ with $z = 48$.

The dual problem is the following (I am using variables s and t for the dual. You could use any two letters, even x and y.) Maximize $w = 4s + 20t$ subject to

$$\begin{aligned} s + 7t &\leq 14 \\ 2s + 6t &\leq 20 \\ 0 &\leq s, t. \end{aligned}$$

The optimal for the dual is $s = 7$ and $t = 1$ with $w = 48$. The dual solution gives the shadow prices for the primal. Since the shadow price for the first constraint is $s = 1$, a change of ϵ in the constraint value would lead to a change of $7 \cdot \epsilon$ in the optimum, namely $48 + 7\epsilon$. □

(11) Do the minimization problem: LEF 9.4, problem 8. *This is the same problem as above. Realized after solving it!*

(12) State the maximum theorem.

Solution. A continuous function on a compact set attains a maximum and a minimum value.

A more elaborate way: Let $S \subset \mathbf{R}^n$ be a compact set and $f : S \to \mathbf{R}$ a continuous function. Then there exists an $a \in S$ such that $f(x) \geq f(a)$ for all $x \in S$, and there exists a $b \in S$ such that $f(x) \leq f(b)$ for all $x \in S$. $\square$

(13) Do LEF 9.3 problem 10. Formulate the dual minimization problem. What is the optimum solution of the dual?

Solution. The dual problem is the following. Minimize $w = 6y_1 + 12y_2$ subjet to

$$\begin{aligned} y_1 + 3y_2 &\geq 1 \\ 2y_1 + 2y_2 &\geq 1 \\ y_1, y_2 &\geq 0. \end{aligned}$$

The solution to the dual is given by the shadow prices for the primal, which we can solve using the simplex method.

I won't write the steps of the simplex method (but you will have to on the exam). The optimal solution turns out to be $x_1 = 3; x_2 = 3/2; y_1 = 1/4; y_2 = 1/4$ with the optimal value $z = w = 9/2$. $\square$

(14) Determine if the following sets are open or closed or neither. Justify your answers.

(a) $S \subset \mathbf{R}^3$ defined by $S = \{\vec{x} \mid 2 < |\vec{x}| < 3\}$.
Solution. S is open. Its the preimage of the open interval $(2, 3)$ under the continuous function $f(\vec{x}) = |\vec{x}|$. $\square$

(b) $S \subset \mathbf{R}^2$ defined by $S = \{(x, y) \mid x^2 + y > 3 \text{ and } x + y < 1\}$.
Solution. S is open. It is the intersection of the set $\{(x, y) \mid x^2 + y > 3\}$ and $\{(x, y) \mid x + y < 1\}$. The first is the preimage of the open set $(3, +\infty)$ under the continuous function $f(x, y) = x^2 + y$. The second is the preimage of the open set $(-\infty, 1)$ under the continuous function $g(x, y) = x + y$. Since intersection of finitely many open sets is open, S is open. $\square$

(c) $S \subset \mathbf{R}$ defined by $S = \{x \mid x = 1/n + 1/m$ for some positive integers m and $n\}$.
Solution. S is not open. To see this, note that $2 = 1/1 + 1/1$ lies in S but no number bigger than 2 lies in S. Since every open interval containing 2 contains points bigger than 2, we conclude that no open interval that contains 2 is a subset of S. In other words, 2 is not an interior point of S.
S is not closed. To see this, note that $0 \notin S$. But 0 is a boundary point of S. Any open interval around 0 contains a point of the form $1/n$ for large enough n, and $1/n = 1/2n + 1/2n$ is a point of S. Hence every open interval around 0 contains a point of S (namely $1/n$) and a point of S^c (namely 0), which makes 0 a boundary point of S. Since 0 is a boundary point of S but not a point of S, we get that S is not closed. $\square$

(15) Let $S \subset \mathbf{R}^n$ be a set. Let S° be the set of interior points of S. Show that S° is an open set. What is S° for $S = [0, 1] \subset \mathbf{R}$? For $S = \{0, 1\} \subset \mathbf{R}$?

Solution. Let x be a point of S°. We have to show that x is an interior point of S°. Since $x \in S$, there exists an $r > 0$ such that the ball $B_r(x)$ is contained in S. But since $B_r(x)$ is open, every $y \in B_r(x)$ is an interior point of $B_r(x)$ and hence an interior point of S. Therefore, $B_r(x) \subset S^\circ$. This shows that x is an interior point of S°.

For $S = [0, 1]$, we have $S^\circ = (0, 1)$. For $S = \{0, 1\}$, we have $S^\circ = \varnothing$. $\square$