AO HW #9

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25/25

①

a) $\int x^2+\dot{x}^2+2xe^t\,dt$

$F(t,x,\dot{x}) = x^2+\dot{x}^2+2xe^t$

$F_1(t,x,\dot{x}) = 2xe^t$, $F_2(t,x,\dot{x}) = 2x+2e^t$, $F_3(t,x,\dot{x}) = 2\dot{x}$

By E-L eq, $2x+2e^t - \frac{d}{dt}(2\dot{x}) = 2x+2e^t - 2\ddot{x} = 0$

$\Rightarrow x+e^t-\ddot{x} = 0$

b) $\int -e^{\dot{x}-x}\,dt$

$F = -e^{\dot{x}-x}$

$F_1 = 0$, $F_2 = e^{\dot{x}-x}$, $F_3 = -e^{\dot{x}-x}$

E-L eq: $e^{\dot{x}-x} - \frac{d}{dt}(-e^{\dot{x}-x}) = e^{\dot{x}-x} + (\ddot{x}-\dot{x})e^{\dot{x}-x} = 0$

$\Rightarrow (1+\ddot{x}-\dot{x})e^{\dot{x}-x} = 0$

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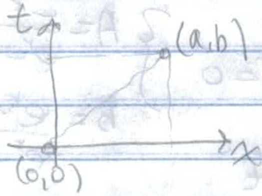
②



length $= \sqrt{\Delta x^2+\Delta t^2} = \sqrt{1+\frac{\Delta t^2}{\Delta x^2}}\cdot\Delta x = \sqrt{1+(t')^2}\,\Delta x$

$\min \int_0^a \sqrt{1+(t')^2}\,dx$ subject to $t(0)=0$ and $t(a)=b$

$F(u,v,w) = \sqrt{1+w^2} \Rightarrow F_2 = 0$, $F_3 = \frac{1}{2}\cdot(1+w^2)^{-\frac{1}{2}}\cdot 2w = w(1+w^2)^{-\frac{1}{2}} = \dot{t}(1+\dot{t}^2)^{-\frac{1}{2}}$

By E-L eq: $0 - \frac{d}{dx}\left(\dot{t}(1+\dot{t}^2)^{-\frac{1}{2}}\right) = -\ddot{t}(1+\dot{t}^2)^{-\frac{1}{2}} + \frac{\dot{t}}{2}(1+\dot{t}^2)^{-\frac{3}{2}}\cdot 2\dot{t}\cdot\ddot{t}$

$\Rightarrow -\ddot{t}(1+\dot{t}^2)^{-\frac{1}{2}} + \dot{t}^2(1+\dot{t}^2)^{-\frac{3}{2}}\ddot{t} = 0$

Straight line $\frac{b}{a}x = t(x)$

$t'(x) = \frac{b}{a}$ $t''(x) = 0$

$\min \int_0^a \sqrt{1+(\frac{b}{a})^2}\,dx = \sqrt{1+(\frac{b}{a})^2}\cdot a = \sqrt{a^2+b^2}$

plug into E-L eq: $-0\cdot(1+(\frac{b}{a})^2)^{-\frac{1}{2}} + (\frac{b}{a})^2\cdot(1+(\frac{b}{a})^2)^{-\frac{3}{2}}\cdot 0 = 0 = 0$ ✓

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③ $\int_1^2 \frac{\dot{x}^2}{t^2}\,dt$ s.t. $x(1)=0$ $x(2)=1$

$F(u,v,w) = \frac{w^2}{u^2}$, $F_2 = 0$, $F_3 = \frac{2}{u^2}w = \frac{2}{t^2}\dot{x} = 2t^{-2}\cdot\dot{x}$

By E-L eq: $0 - \frac{d}{dt}(2t^{-2}\cdot\dot{x}) = 0 \Rightarrow \frac{d}{dt}(2t^{-2}\dot{x}) = 0$

$\Rightarrow 2t^{-2}\dot{x} = C$

$\Rightarrow \dot{x} = \frac{C}{2}t^2 \Rightarrow x = \frac{C}{6}t^3 + D$

Since $x(1)=0$, $x(2)=1$

$\left.\begin{aligned} \frac{1}{6}C+D=0 &\Rightarrow C+6D=0 \\ \frac{4}{3}C+D=1 &\Rightarrow 4C+3D=3 \end{aligned}\right\} \Rightarrow C=\frac{6}{7}, D=-\frac{1}{7}$

$\Rightarrow x = \frac{1}{7}t^3 - \frac{1}{7}$

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④ max $\int_0^1 e^{-rt}\ln C\,dt$ s.t. $k(0)=k_0$ and $k(1)=0$

a) $F(u,v,w)=e^{-ru}\ln(cv-w)$

$F_2 = e^{-ru}\cdot(cv-w)^{-1}\cdot c = ce^{-rt}(ck-\dot{k})^{-1}$

$F_3 = e^{-ru}\cdot(cv-w)^{-1}(-1) = -e^{-rt}(ck-\dot{k})^{-1}$

E-L eq: $ce^{-rt}(ck-\dot{k})^{-1} + \frac{d}{dt}\left(e^{-rt}(ck-\dot{k})^{-1}\right)=0$

$$ce^{-rt}(ck-\dot{k})^{-1} - re^{-rt}(ck-\dot{k})^{-1} - e^{-rt}(ck-\dot{k})^{-2}\cdot(c\dot{k}-\ddot{k})=0$$

$$c(ck-\dot{k})^{-1} - r(ck-\dot{k})^{-1} - (ck-\dot{k})^{-2}(c\dot{k}-\ddot{k})=0$$

$$-(c\dot{k}-\ddot{k})+c(ck-\dot{k})-r(ck-\dot{k})=0$$

$$\ddot{k}-c\dot{k}+c^2k-c\dot{k}-rck+r\dot{k}=0$$

$$\ddot{k}-(2c-r)\dot{k}+(c^2-rc)k=0$$

$$\ddot{k}-(2c-r)\dot{k}+c(c-r)k=0$$

$\Rightarrow k(t)=Ae^{ct}+Be^{(c-r)t}$

$k(0)=A+B=k_0$

$k(1)=Ae^{c}+Be^{c}\cdot e^{-r}=0 \Rightarrow A+Be^{-r}=0$

$\Bigg\}\ A=\frac{-k_0}{e^r-1}$, $B=\frac{e^r\cdot k_0}{e^r-1}$

$$\Rightarrow k(t)=\left(\frac{e^r\cdot k_0}{e^r-1}\right)e^{(c-r)t}+\left(\frac{-k_0}{e^r-1}\right)e^{ct}$$

b) $c=0.1$ and $k_0=1$ 5

then $k(t)=\frac{e^r}{e^r-1}\cdot e^{(0.1-r)t}-\frac{1}{e^r-1}e^{0.1t}$

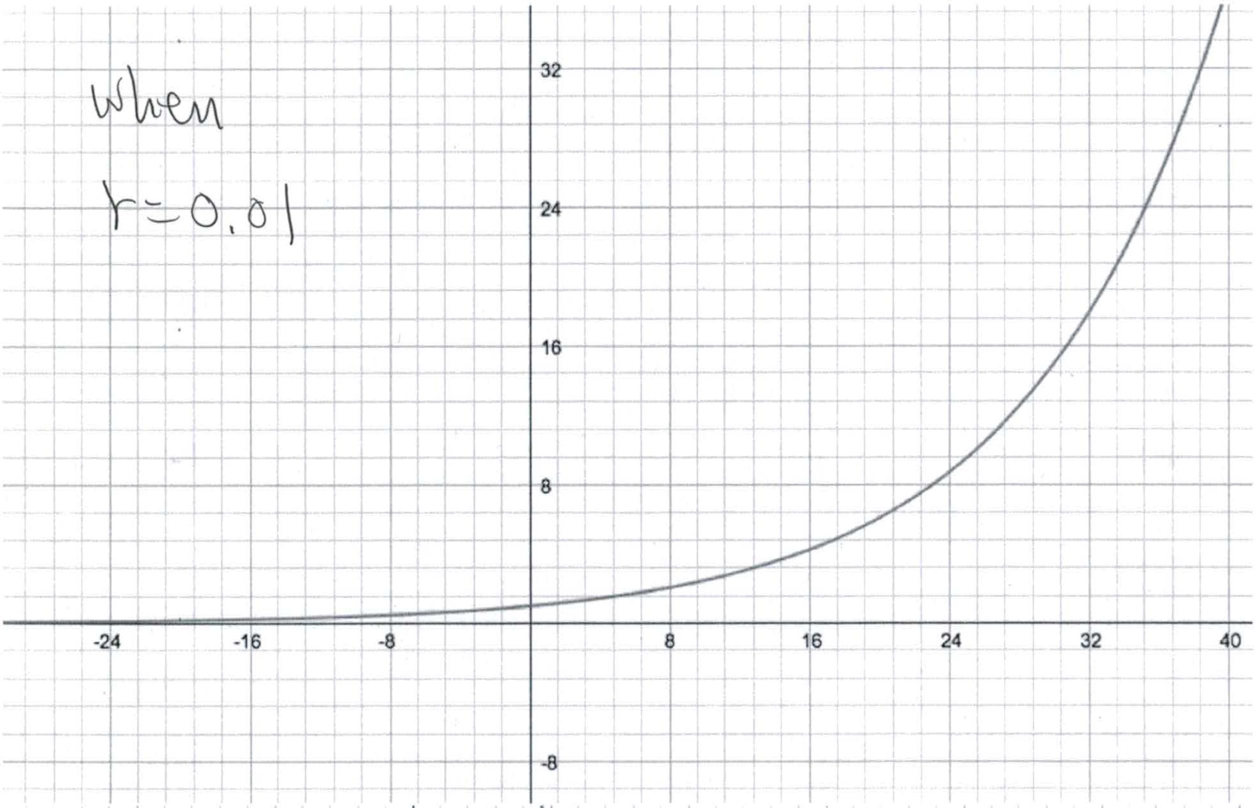
$$k'(t)=\frac{(0.1-r)e^r\cdot e^{(0.1-r)t}}{e^r-1}-\frac{0.1}{e^r-1}e^{0.1t}$$

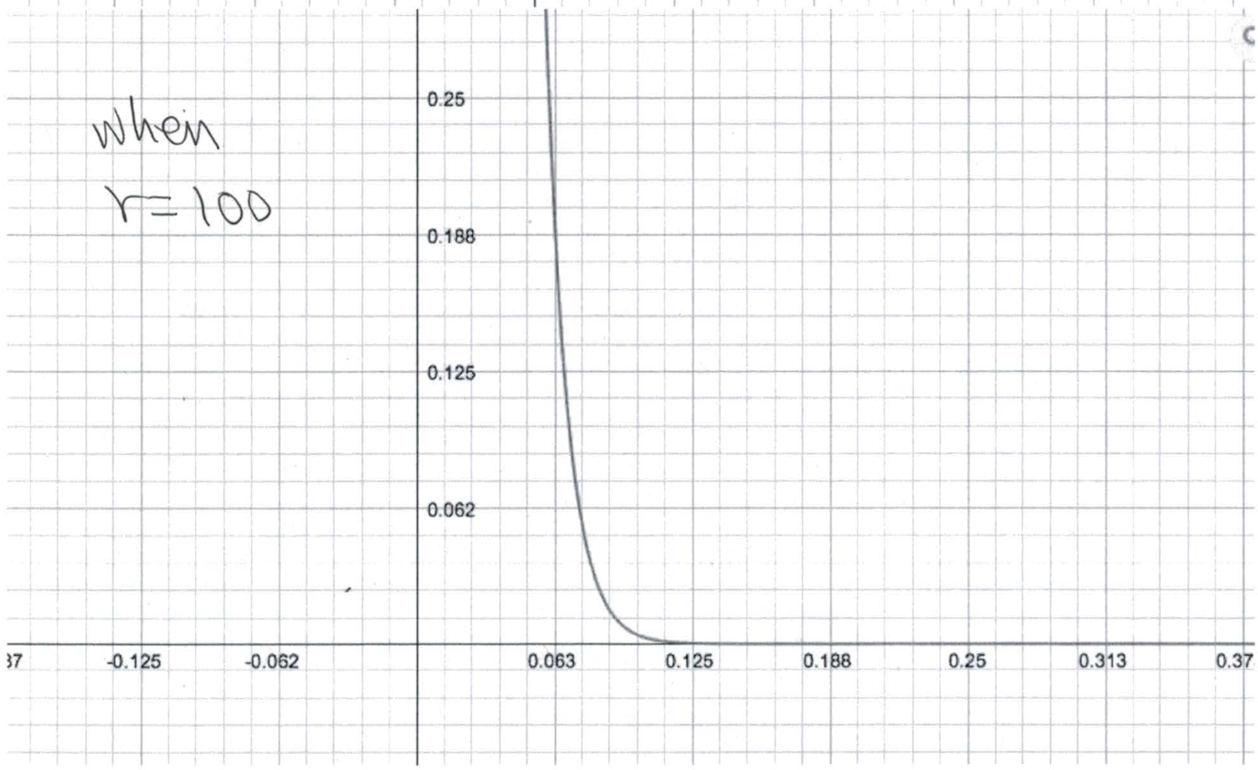
$$C=ck-k'=0.1k-k'=\frac{0.1e^r\cdot e^{(0.1-r)t}}{e^r-1}-\frac{0.1}{e^r-1}e^{0.1t}+\frac{0.1}{e^r-1}e^{0.1t}-\frac{(0.1-r)e^r\cdot e^{(0.1-r)t}}{e^r-1}$$

$$=\frac{0.1e^r\cdot e^{(0.1-r)t}}{e^r-1}-\frac{(0.1-r)e^r\cdot e^{(0.1-r)t}}{e^r-1}$$

$$=\frac{r\cdot e^r\cdot e^{(0.1-r)t}}{e^r-1}$$

$\Rightarrow C(t)=\frac{r\cdot e^r\cdot e^{(0.1-r)t}}{e^r-1}$ when $c=0.1$ and $k_0=1$


when
r=0.01
32
24
16
8
-8
-24
-16
-8
8
16
24
32
40


when
r=100
0.25
0.188
0.125
0.062
-0.125
-0.062
0.063
0.125
0.188
0.25
0.313
0.37

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(5) $\min \int (T-V)\,dt \qquad V(h) = mgh, \quad T(v) = \frac{1}{2}mv^2$

We know that $\dot{h} = v$ and $\ddot{h} = a$

Then $V(h) = mgh, \quad T(\dot{h}) = \frac{1}{2}m\dot{h}^2$

$F(t, h, \dot{h}) = \frac{1}{2}m\dot{h}^2 - mgh$

$F_2 = -mg, \quad F_3 = m\dot{h}$

By E-L eq: $-mg - \frac{d}{dt}(m\dot{h}) = 0$

$$-mg - m\ddot{h} = 0 \implies mg + m\ddot{h} = 0$$

$$\implies -mg = m\ddot{h}$$

and since $\ddot{h} = a$, $\underset{F_2}{\overset{\parallel}{-mg}} = \underset{F}{\overset{\parallel}{ma}}$

Force is the negative of the derivative of potential energy.

$F = -\frac{d}{dt}V = -\frac{d}{dt}(mgh) = -mg. \implies F = -mg.$

?

$\partial/\partial t(mgh) = mg\dot{h}$?

mean $\partial/\partial h$?

$$= m\ddot{h}$$
$$= ma$$
$$\implies F = ma$$

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