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Analysis & Opt MW840

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excellent

(1) Maximize $f(\vec{x}) = 1-(x-1)^2 - e^{y^2}$

subject to $g(\vec{x}) = x^2+y^2-1 \leq 0$

- by Value theorem, a continuos function will achieve a min and max on a compact set

- $2(x-1) = \lambda(2x)$
- $-2ye^{y^2} = \lambda(2y)$

if $g(x)$ slack, then $\lambda=0$ $\therefore (x,y)=(1,0)$

Check CQ: holds since there are no vectors

if $g(x)$ binding, $x^2+y^2=1$, $\lambda \geq 0$

Check CQ: holds since only one vector $\therefore$ linearly indep

$\therefore$ have: $x-1=\lambda x$

$-2ye^{y} = \lambda 2y \Rightarrow y=0$ or $\underbrace{e^{y}=-\lambda}_{\text{can't be true if } \lambda\geq 0 \; \therefore \text{ discard}}$

$x^2+y^2=1$

$\therefore$ Only pt is $(1,0)$, $\lambda=0$

$\therefore$ max of 0 at $(1,0)$

note analytically that 0 is the maximum value of $-(x-1)^2$ and -1 is the maximum value of $-e^{y^2}$, these are both constantly decreasing functions, $\therefore$ we know 0 at $(1,0)$ is the maximum value of $1-(x-1)^2+e^{y^2}$ as this is the composition of 3 functions that are all achieving their maximum values

(2) maximize $x^{\alpha}y^{\beta}$

Sub to $g_1(x,y) = x+y-2 \leq 0$

$g_2(\vec{x}) = -x \leq 0$

$g_3(\vec{y}) = -y \leq 0$

Set is compact $\cap$ function continuos $\Rightarrow$ have max and min

- $\alpha x^{\alpha} y^{\beta} = \lambda_1 - \lambda_2$
- $\beta x^{\alpha} y^{\beta} = \lambda_1 - \lambda_3$

if all three binding, CQ does not hold, also impossible since $0 \neq 2$

if 2 and 3 slack, CQ holds, have:

- $\alpha x^{\alpha-1}y^{\beta} = \lambda_1$
- $\beta x^{\alpha}y^{\beta-1} = \lambda_1$
- $x+y=2$

$\Rightarrow \frac{\alpha}{\beta}x^{-1}y = 1 \Rightarrow \frac{x}{\alpha} = \frac{y}{\beta}$

$x+y=2 \Rightarrow x+\frac{\beta}{\alpha}x = 2 \Rightarrow x = 2\cdot\frac{\alpha}{\alpha+\beta}$

$y = 2\cdot\frac{\beta}{\alpha+\beta}$

$f(\vec{x}) = \left(2\frac{\alpha}{\alpha+\beta}\right)^{\alpha}\left(2\frac{\beta}{\alpha+\beta}\right)^{\beta} > 0$

if 2 binding, 1 binding, 3 slack: {CQ holds} $\langle 1,1\rangle$, $\langle -1,0\rangle$ LI ✓

- $\alpha x^{\alpha}y^{\beta} = \lambda_1 - \lambda_2$
- $\beta x^{\alpha}y^{\beta} = \lambda_1$
- $x+y=2$

$\Rightarrow \lambda_1=\lambda_2=0$

$x=0$

$y=2$

$\lambda_3=0$

$f(\vec{x})=0$

if 3 binding, 1 binding, 2 slack: {CQ holds} $\langle 1,1\rangle$, $\langle 0,-1\rangle$ LI ✓

$x=2$

$y=0$

$\lambda_1=\lambda_2=\lambda_3=0$

$f(\vec{x})=0$

if 2 and 3 binding, 1 slack:

$x=0$, $y=0$, $\lambda_1=\lambda_2=\lambda_3$, $f(\vec{x})=0$

maximum value of $2^{\alpha+\beta}\left(\frac{\alpha}{\alpha+\beta}\right)^{\alpha}\left(\frac{\beta}{\alpha+\beta}\right)^{\beta}$ at $\left(\frac{2\alpha}{\alpha+\beta}, \frac{2\beta}{\alpha+\beta}\right)$

(2) Envelope Theorem

$$\frac{\partial f^*(\vec{v})}{\partial v_i} = \frac{\partial L(\vec{x},\vec{v})}{\partial v_i}$$

$\Rightarrow$ Show

$$\frac{\partial f}{\partial \alpha} = \frac{\partial L}{\partial \alpha}$$ {this will be extended to $\frac{\partial f}{\partial \beta}$ by symmetry}

$$L = x^\alpha y^\beta - \lambda_1(x+y-2) \quad \therefore \quad L = x^\alpha y^\beta - \alpha x^{\alpha-1} y^\beta (x+y-2)$$

$$\frac{\partial L}{\partial \alpha} = \ln x \, x^\alpha y^\beta - (x+y-2) y^\beta (x^{\alpha-1} + \alpha x^{\alpha-1} \ln x)$$

$$\Rightarrow \left.\frac{\partial L}{\partial \alpha}\right|_{x^*, y^*} = \ln\left(\frac{2\alpha}{\alpha+\beta}\right)\left(\frac{2\alpha}{\beta+\alpha}\right)^\alpha \left(\frac{2\beta}{\beta+\alpha}\right)^\beta$$

$$f^*(\vec{v}) = 2^{\alpha+\beta} \alpha\beta \left(\frac{1}{\beta+\alpha}\right)^{\alpha+\beta}$$

$$\frac{\partial f^*}{\partial \alpha} = (2\beta)^\beta \cdot \left(\frac{1}{\beta+\alpha}\right)^\beta \cdot (2\alpha)^\alpha \cdot \left(\frac{1}{\beta+\alpha}\right)^\alpha \cdot \left(\ln 2 + \ln\left(\frac{1}{\alpha+\beta}\right) - 1 + \ln\alpha + 1\right)$$

$$= \ln\left(\frac{2\alpha}{\alpha+\beta}\right)\left(\frac{2\alpha}{\beta+\alpha}\right)^\alpha \left(\frac{2\beta}{\alpha+\beta}\right)^\beta = \frac{\partial L}{\partial \alpha}$$

$$\therefore \frac{\partial f^*(\vec{v})}{\partial \alpha} = \frac{\partial L(\vec{x},\vec{v})}{\partial \alpha}$$

and by extension thru symmetry this is true for β

$$\therefore \frac{\partial f^*}{\partial v_i} = \frac{\partial L}{\partial v_i}$$

(3) Minimize $x^2+y^2 \Rightarrow \max -x^2-y^2$

s.t. $y^2-(x-1)^3 \le 0$

analytically, min probably at $(1,0)$

$-2x = \lambda 3(x-1)^2$

$-2y = \lambda 2y \Rightarrow y=0$ OR $\lambda = -1$

if slack, $\lambda = 0 \Rightarrow y=0, x=0$
which is not a pt in the set $\therefore$ discard

if binding,
$y^2 = (x-1)^3$

$\therefore$ if $y=0, x=1 \Rightarrow$ pt $(1,0)$ ↳ but λ is unfindable!

if $y \neq 0$, $\lambda = -1$ $\therefore$ discard

$\therefore$ ~~$2x = 3(x-1)^2$~~

~~$\Rightarrow 0 = 3x^2 - 6x + 1 - 2x$~~

~~$\Rightarrow 3x^2 - 8x + 1 = 0$~~ irrelevant

{note that $(x-1)^3 \ge 0$ since it is greater than y^2}

the lagrange mult doesn't exist because the CQ fails
∇g binding $= \langle 0,0 \rangle$

maximum at pt at $(1,0)$

$\therefore$ minimum of x^2+y^2 of 1 at $(1,0)$

(4) Maximize $11A + 16B + 15C$

s.t. $A + 2B + \frac{3}{2}C \le 120$

$\frac{2}{3}A + \frac{2}{3}B + C \le 46$

$\frac{1}{2}A + \frac{1}{3}B + \frac{1}{2}C \le 24$

$A=6,\ B=51,\ C=8$

$\Rightarrow 6 + 102 + 12 = 120$ binding

$\Rightarrow 4 + \frac{102}{3} + 8 = 56$ binding

$\Rightarrow 3 + \frac{51}{3} + 4 = 24$ binding

CQ satisfied as the 3 vectors are linearly independent

$$\left.\begin{aligned} 11 &= \lambda_1 + \tfrac{2}{3}\lambda_2 + \tfrac{1}{2}\lambda_3 \\ 16 &= 2\lambda_1 + \tfrac{2}{3}\lambda_2 + \tfrac{1}{3}\lambda_3 \\ 15 &= \tfrac{3}{2}\lambda_1 + \lambda_2 + \tfrac{1}{2}\lambda_3 \end{aligned}\right\} \quad \begin{aligned} \lambda_1 &= 6 \\ \lambda_2 &= 3 \\ \lambda_3 &= 6 \end{aligned} \quad \therefore \lambda_i \ge 0$$

$\frac{\partial L}{\partial r_i} = -\lambda_j x_h^*$ where j and h represent the corresponding indices

(a) $\lambda_j = \lambda_2 = 3$
$x_h = A$ $\Rightarrow \frac{\partial L}{\partial(\text{trim } A)} = -3A^* = -18$

$\therefore$ profit increases by 18£
since by envelope thm $\frac{\partial L}{\partial r_i} = \frac{\partial f^*}{\partial r_i}$

(b) $\lambda_j = 6 = \lambda_3$
$x_h = C \Rightarrow \frac{\partial L}{\partial(\text{pack } C)} = -6C^* = -48$

$\therefore$ profit increases by 48£

Prove that this point is a global maximum: this point is a local max because the Langrangian is concave, also, this set is bounded and closed $\therefore$ compact and as such our continuos function will attain a maximum and minimum by extreme value theorem

(5) maximize $cx+y$

s.t. $x^2+2y^2=2$

$g_1(\vec{x})=-x\le 0$

$g_2(\vec{x})=-y\le 0$

Analyze

~~Know that if $c\ge 1$, $x=\sqrt{2}$, $f^*(c)=\sqrt{2}c$, $y=0$~~

if $c\le 0$, then $y=1$, $x=0$, $f^*(c)=1$

e.g.

~~if $c=1/2$, maximize $f(x,y)=\frac{1}{2}x+y$ s.t. $x^2+2y^2=2$, $x\ge 0, y\ge 0$~~

take $c>0$

~~take $0<c<1$~~

$$C=\lambda_1 2x-\lambda_2$$
$$1=4y\lambda_1-\lambda_3$$
$$x^2+2y^2=2$$

if y binding, $\lambda_3=-1<0$ $\therefore$ discard

Same for if x is binding

take $y\ge 0$ and $x\ge 0$ slack

have:

$$\begin{aligned} c&=\lambda_1 2x \\ 1&=\lambda_1 4y \end{aligned} \Rightarrow c=\frac{x}{2y}\Rightarrow x=2cy$$

$$x^2+2y^2=2$$
$$4c^2y^2+2y^2=2$$
$$y^2=\frac{2}{4c^2+2}$$
$$x^2=\frac{8c^2}{4c^2+2}$$

$$\therefore f^*(c)=c^2\sqrt{\frac{8}{4c^2+2}}+\sqrt{\frac{2}{4c^2+2}}$$
$$=\sqrt{\frac{2}{4c^2+2}}\cdot(2c^2+1)\quad \text{for } c>0$$

$$f^*(c)=\begin{cases}1 & \text{for } c\le 0\\ \sqrt{2c^2+1} & \text{for } c>0\end{cases}$$