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1. Max e^x+y+z subject to ① $x+y+z=1$ $(g_1(x,y,z))$

② $x^2+y^2+z^2=1$ $(g_2(x,y,z))$

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excellent

$\mathcal{L}(x,y,z) = e^x+y+z-\lambda g_1(x,y,z)-\lambda g_2(x,y,z)$

$\frac{\partial \mathcal{L}(x,y,z)}{\partial x} = e^x - \lambda_1\cdot(1) - \lambda_2\cdot 2x = 0$

$\Rightarrow e^x = \lambda_1 + 2\lambda_2 x$

$\frac{\partial \mathcal{L}(x,y,z)}{\partial y} = 1-\lambda_1\cdot 1-\lambda_2 2y = 0$

$\Rightarrow 1 = \lambda_1 + 2\lambda_2 y$

$\frac{\partial \mathcal{L}(x,y,z)}{\partial z} = 1-\lambda_1 - 2\cdot\lambda_2 z = 0$

$\Rightarrow 1 = \lambda_1 + 2\lambda_2 z$

$\Rightarrow \begin{cases} e^x = \lambda_1+2\lambda_2 x \\ 1 = \lambda_1+2\lambda_2 y \\ 1 = \lambda_1+2\lambda_2 z \end{cases} \Rightarrow \lambda_1+2\lambda_2 y = \lambda_1+2\lambda_2 z$

if $\lambda_2=0$. $\lambda_1=1$ $e^x=1$ $x=0$. $0+y+z=1$ $0+y^2+z^2=1$.

$(1-z)^2+z^2 = 1+z^2-2z+z^2=1$ $2z^2=2z$ $z^2=z$ $z=0$ or $z=1$

if $z=0$, $y=1$, if $z=1$, $y=0$.

if $\lambda_2\neq 0$, $y=z$. $x+2y=1$ $x^2+2y^2=1$

$x^2=(1-2y)^2=1+4y^2-4y=1-2y^2$. $\Rightarrow$ $6y^2=4y$ $3y^2=2y$

$y=0$ or $y=\frac{2}{3}$.

if $y=0, \Rightarrow z=0$, $x=1$,

$\lambda_1=1$, $e=1+2\lambda_2$

$\lambda_2=\frac{e-1}{2}$.

if $y=\frac{2}{3}, \Rightarrow z=\frac{2}{3}$. $x=-\frac{1}{3}$

$\lambda_1+\frac{4}{3}\lambda_2=1$

$e^{-\frac{1}{3}}=\lambda_1-\frac{2}{3}\lambda_2$

Possible maximum point: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \underbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{pmatrix}}$.

$e^0 + 1 + 0 = 2$

$e^0 + 0 + 1 = 2$

$e^1 + 0 + 0 = e$

$e^{-\frac{1}{3}} + \frac{2}{3} + \frac{2}{3} \approx 2.049$.

$\therefore$ the maximum point is $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

corresponding $\lambda_1 = 1$

$\lambda_2 = \frac{e-1}{2}$

If the constraints are changed. λ_1, λ_2 are the corresponding changing rate.

$\therefore \Delta max = \Delta_1 \cdot \lambda_1 + \Delta_2 \cdot \lambda_2 = 0.02 \cdot 1 - 0.02 \cdot \frac{e-1}{2} = 0.02 \left(\frac{2-e+1}{2}\right)$

$\therefore$ max value $= e + 0.02 \cdot \frac{3-e}{2} = \underline{e + 2.8172 \times 10^{-3}}$

(2). Max $1 - x^2 - y^2$ subject to $x + y = m$.

let $g(x) = x + y - m$.

$\mathcal{L}(x,y) = 1 - x^2 - y^2 - \lambda g(x)$

$\frac{\partial \mathcal{L}(x,y)}{\partial x} = -2x - \lambda_1 \cdot 1 = 0 \Rightarrow -2x = \lambda_1$

$\frac{\partial \mathcal{L}(x,y)}{\partial y} = -2y - \lambda_1 \cdot 1 = 0 \Rightarrow -2y = \lambda_1$

$\begin{cases} -2x = \lambda_1 \\ -2y = \lambda_1 \\ x + y - m = 0 \end{cases} \Rightarrow x = y$.

$\therefore 2x = m \quad x = \frac{m}{2}. \quad y = \frac{m}{2}. \quad \lambda_1 = -m$.

Max value $\Rightarrow 1 - \left(\frac{m}{2}\right)^2 - \left(\frac{m}{2}\right)^2 = 1 - \frac{m^2}{2}$.

Now assume that max value as a function of m.

$f(x^*) = 1 - \frac{m^2}{2}$

$\frac{\partial f(x^*)}{\partial m} = -m = \lambda_1$

$\therefore$ the rate of change is equal to the Lagrange multiplier.

(3). Max x^2+y^2 s.t. $2x^2+y^2=2$

let $g(x,y)=2x^2+y^2-2$.

$\mathcal{L}(x,y)=x^2+y^2-\lambda_1 g(x,y)$

$$\frac{\partial \mathcal{L}(x,y)}{\partial x}=2x-\lambda_1\cdot 4x=0$$

$$\frac{\partial \mathcal{L}(x,y)}{\partial y}=2y-\lambda_1\cdot 2y=0.$$

$$\Rightarrow \begin{cases} 2x=4\lambda_1 x \\ 2y=2\lambda_1 y \\ 2x^2+y^2-2=0 \end{cases} \Rightarrow \begin{matrix} x=2\lambda_1 x \\ y=\lambda_1 y \end{matrix}$$

If $x=0$, $y^2=2$ $y=\pm\sqrt{2}$. if $y=\sqrt{2}$, $\lambda_1=1$.
if $y=-\sqrt{2}$, $\lambda_1=1$.

If $y=0$, $2x^2=2$ $x=\pm 1$ if $x=1$, $\lambda_1=\frac{1}{2}$
$x=-1$, $\lambda_1=\frac{1}{2}$.

If $x\neq 0$, $y\neq 0$. $\lambda_1=\frac{1}{2}$ and $\lambda_1=1$, descard.

$$\therefore \begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ \sqrt{2} \end{pmatrix}, \begin{pmatrix} 0 \\ -\sqrt{2} \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix}.$$

Second Order Condition:

$$\frac{\partial g_1}{\partial x}=4x \quad \frac{\partial g_1}{\partial y}=2y. \quad \frac{\partial f}{\partial x^2}=2 \quad \frac{\partial f}{\partial^2 y}=2 \quad \frac{\partial f}{\partial x\partial y}=0.$$

$\therefore$ Bordered matrix:

$$\left(\begin{array}{c|cc} 0 & 4x & 2y \\ \hline 4x & 2-4\lambda_1 & 0 \\ 2y & 0 & 2-2\lambda_1 \end{array}\right)$$

①. $(x\ y)=(0\ \sqrt{2})$ $B_2=2\sqrt{2}\cdot[-(2\sqrt{2}\cdot(-2))]=16.$
$(-1)^2\cdot B_2>0$. $\therefore$ neg. definite.
$(0\ \sqrt{2})$ is a local max

②. $(x\ y)=(0,-\sqrt{2})$ $B_2=-2\sqrt{2}\cdot(-4\sqrt{2})=16.$
$(-1)^2\cdot B_2>0$ $\therefore$ neg. definite. $(0,-\sqrt{2})$ is local max

③ $(x\ y)=(1\ 0)$ $B_2=-16$
$(-1)^1\cdot B_2>0$ $\therefore$ pos. definite $(1\ 0)$ is local min

④ $(x\ y)=(-1\ 0)$ $(-1)^1\cdot B_2>0$ $\therefore$ pos def. $(-1\ 0)$ is local min.

Case 6: ①. slack ②. binding ③ slack

$\lambda_1 = \lambda_3 = 0$. $x = 0$

$\frac{\partial L(x,y)}{\partial x} = y + \lambda_2 = 0$ $\frac{\partial L(x,y)}{\partial y} = x = 0$. $\Rightarrow y = \lambda_2 = 0 \Rightarrow$ discard.

Case 7: ① slack ② slack ③ binding

$\lambda_1 = \lambda_2 = 0$. $y = 0$. $\frac{\partial L(x,y)}{\partial x} = y = 0$ $\frac{\partial L(x,y)}{\partial y} = x + \lambda_3 = 0 \Rightarrow$ discard

Case 8: ① slack ② binding ③ binding

$\Rightarrow x = y = 0$.

(4). Max $x+y+z$ st $g_1(x,y,z) = x^2+y^2+z^2-1$.

$g_2(x,y,z) = x-y-z-1$.

$L(x,y,z) = x+y+z - \lambda g_1(x,y,z) - \lambda_2 g_2(x,y,z)$

$$\frac{\partial L(x,y,z)}{\partial x} = 1 - \lambda_1 \cdot 2x - \lambda_2 \cdot 1 = 0.$$

$$\frac{\partial L(x,y,z)}{\partial y} = 1 - 2y \cdot \lambda_1 - \lambda_2 \cdot (-1) = 0$$

$$\frac{\partial L(x,y,z)}{\partial z} = 1 - \lambda_1 \cdot 2z - \lambda_2 \cdot (-1) = 0.$$

$$\begin{cases} 1 = 2\lambda_1 x + \lambda_2 \\ 1 = 2\lambda_1 y - \lambda_2 \\ 1 = 2\lambda_1 z - \lambda_2 \\ x^2+y^2+z^2-1 = 0. \\ x-y-z-1 = 0 \end{cases} \Rightarrow. \quad \begin{aligned} 2\lambda_1 y - \lambda_2 &= 2\lambda_1 z - \lambda_2 \\ \lambda_1 y &= \lambda_1 z. \end{aligned}$$

If $\lambda_1 \neq 0$. $y = z$, $x = 1+2y$. $(1+2y)^2 + 2y^2 = 1$. $1+4y^2+4y+2y^2=1$

$6y^2 = -4y$ $y = 0$ or $y = -\frac{2}{3}$ if $y=0, \Rightarrow z=0$, $x=1$.

if $y = -\frac{2}{3} \Rightarrow z = -\frac{2}{3}$ $x = -\frac{1}{3}$

If $\lambda_1 = 0$, $1 = \lambda_2$ $1 = -\lambda_2 \Rightarrow \lambda_2$ doesn't exist.

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \text{ or } \begin{pmatrix} -\frac{1}{3} \\ -\frac{2}{3} \\ -\frac{2}{3} \end{pmatrix}$$

$\frac{\partial g_1}{\partial x} = 2x \quad \frac{\partial g_1}{\partial y} = 2y \quad \frac{\partial g_1}{\partial z} = 2z$

$\frac{\partial g_2}{\partial x} = 1 \quad \frac{\partial g_2}{\partial y} = -1 \quad \frac{\partial g_2}{\partial z} = -1$

$\therefore$ matrix for the constraint is $\begin{bmatrix} 2x & 2y & 2z \\ 1 & -1 & -1 \end{bmatrix}$

$\frac{\partial^2 L}{\partial^2 x} = -2\lambda_1 \quad \frac{\partial^2 L}{\partial^2 y} = -2\lambda_1 \quad \frac{\partial^2 L}{\partial^2 z} = -2\lambda_1$

$\frac{\partial L}{\partial x \partial y} = 0 \quad \frac{\partial L}{\partial x \partial z} = 0 \quad \frac{\partial L}{\partial y \partial z} = 0$

Bordered matrix:

$$\left(\begin{array}{cc|ccc} 0 & 0 & 2x & 2y & 2z \\ 0 & 0 & 1 & -1 & -1 \\ \hline 2x & 1 & -2\lambda_1 & 0 & 0 \\ 2y & -1 & 0 & -2\lambda_1 & 0 \\ 2z & -1 & 0 & 0 & -2\lambda_1 \end{array}\right) \quad \lambda_1 = 1$$

If $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \Longrightarrow$ Bordered matrix $\left(\begin{array}{cc|ccc} 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 \\ \hline 2 & 1 & -2 & 0 & 0 \\ 0 & -1 & 0 & -2 & 0 \\ 0 & -1 & 0 & 0 & -2 \end{array}\right)$

$m = 2 \quad n = 3.$

$\therefore B_3 = -16. \quad (-1)^3 \cdot B_3 > 0. \quad \therefore$ neg. definite

$(x\ y\ z) = (1\ 0\ 0)$ is a local max

If $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -\frac{1}{3} \\ \frac{2}{3} \\ -\frac{2}{3} \end{pmatrix} \Rightarrow$ Bordered matrix $\left(\begin{array}{cc|ccc} 0 & 0 & -\frac{2}{3} & -\frac{4}{3} & -\frac{4}{3} \\ 0 & 0 & 1 & -1 & -1 \\ -\frac{2}{3} & 1 & 2 & 0 & 0 \\ -\frac{4}{3} & -1 & 0 & 2 & 0 \\ -\frac{4}{3} & -1 & 0 & 0 & 2 \end{array}\right)$

$B_3 = 16. \quad (-1)^2 B_3 > 0.$

$\therefore$ pos definite

$(-\frac{1}{3}, +\frac{2}{3}, -\frac{2}{3})$ is a local min.

(5). Max xy subject to $x+y^2-2 \le 0$

$-x \le 0$

$-y \le 0$.

$x^2+y^2-2 \le 0 \Rightarrow x+y^2-2<0 \cup x^2+y^2-2=0$.

Let ① $g_1 = x+y^2-2$.

② $g_2 = -x$

③ $g_3 = -y$.

$L(x,y) = xy - \lambda_1 g_1 - \lambda_2 g_2 - \lambda_3 g_3$

$\lambda_1 \ge 0$ and $\lambda_1 = 0$ if $g_1 < 0$.

$\lambda_2 \ge 0$ and $\lambda_2 = 0$ if $g_2 < 0$

$\lambda_3 \ge 0$ and $\lambda_3 = 0$ if $g_3 < 0$.

KKT Condition:

$y = \lambda_1 - \lambda_2$

$x = 2\lambda_1 y - \lambda_3$

Case 1: $x+y^2=2$

If $x=0$, $y\ne 0$. $\lambda_3=0$. $y=\lambda_1-\lambda_2$ $2\lambda_1 y=0 \Rightarrow$ discard.

If $x\ne 0$, $y=0$ $\lambda_2=0$ $y=\lambda_1$, $x=\lambda_3 \Rightarrow \lambda_1=0 \Rightarrow$ discard.

If $x=0$, $y=0 \Rightarrow$ discard

If $x\ne 0$, $y\ne 0$ $\lambda_2=\lambda_3=0$ $\therefore y=\lambda_1$ $x=2\lambda_1^2$

$\therefore 3\lambda_1^2=2$ $\lambda_1^2=\frac{2}{3}$ $\lambda_1=\sqrt{\frac{2}{3}}$ $y=\sqrt{\frac{2}{3}}$ $x=\frac{4}{3}$.

Case 2: $x+y^2<2$.

If $x=0$, $y=0$. $\lambda_1=0$. $y=-\lambda_2=0 \Rightarrow$ discard.

If $x\ne 0$, $y=0$. $\lambda_1=\lambda_2=0$ $y=0$, $x=-\lambda_3 \Rightarrow$ discard.

If $x=0$, $y\ne 0$, $\lambda_3=0$. $\lambda_1=0$. $y=-\lambda_2 \Rightarrow$ discard.

If $x\ne 0$, $y\ne 0$, $\lambda_1=\lambda_2=\lambda_3=0$. $y=0 \Rightarrow$ discard

Consider when Constraint Qualification fails.

$\nabla g_1 = (2 \;\; 2y)$ If ①② binding ③ slack, $y=\sqrt{2}$, lin. indep.

$\nabla g_2 = (1 \;\; 0)$ ①③ binding ② slack, ∇g_1 & ∇g_3 lin. indep.

$\nabla g_3 = (0 \;\; -1)$. ②③ binding ① slack, $y=0$, automatically lin indep.

$\nabla g_1 \neq 0$, $\nabla g_2 \neq 0$, $\nabla g_3 \neq 0$

$\therefore$ CQ always holds.

$\therefore$ Max $xy \Rightarrow$ when $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{4}{3} \\ \sqrt{\frac{2}{3}} \end{pmatrix}$

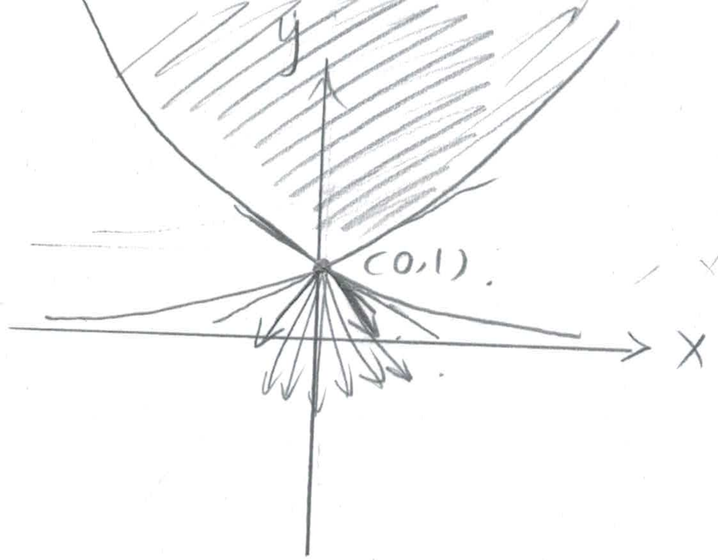
(6). $y \geq e^x \Rightarrow -y \leq -e^x \Rightarrow g_1(x,y) = e^x - y$

$y \geq e^{-x} \quad\; -y \leq -e^{-x} \quad\; g_2(x,y) = e^{-x} - y$.

$\nabla g_1(x,y) = (e^x, -1)$ $\quad e^x \neq 0 \;\; -e^{-x} \neq 0$

$\nabla g_2(x,y) = (-e^{-x}, -1)$ $\quad$ and $e^x \neq e^{-x}$ for

$\therefore$ two constraints satisfy constraint qualification



At $(0,1)$, $\nabla g_1(x,y) = (1,-1)$

$\nabla g_2(x,y) = (-1,-1)$.

$\nabla f(x,y) = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2$, $\lambda_1, \lambda_2 \geq 0$.

$\therefore \nabla f(x,y)$ is a linear combination of ∇g_1 and ∇g_2.

$\therefore$ the possible tips is between $(1,-1)$ and $(-1,-1)$

(7). The preimage of $Q(\vec{x})=1$ is closed, $\therefore Q^{-1}(\vec{x}) = \vec{x}$ is closed.

$Q(\vec{x}) = x^T A x$ according to spectral theorem, there exists orthogonal P such that $A = P^T D P$, D is diagonal.

$\therefore Q(\vec{x}) = x^T P^T D P x$. Let $y = Px$ $\therefore Q(\vec{x}) = y^T P \cdot y$

$= \lambda_1 y_1^2 + \cdots + \lambda_n y_n^2 = 1$.

Then $\lambda_1 P^2 x_1^2 + \cdots + \lambda_n P^2 x_n^2 = 1$,

$\therefore S = \{\vec{x} \mid Q(\vec{x}) = 1\}$ is bounded

$\therefore S$ is compact.

Minimize $x_1^2 + \cdots + x_n^2$ subject to $Q(\vec{x}) = 1$.

$\mathcal{L}(\vec{x}) = x_1^2 + \cdots + x_n^2 - \lambda_1 [Q(\vec{x}) - 1]$

$\frac{\partial \mathcal{L}(\vec{x})}{\partial x_i} = 2x_i - \lambda_1 \cdot 2A x_i$

$\therefore \nabla \mathcal{L} = 2\bar{x} - 2\lambda_1 A\bar{x} = 0$

$\Rightarrow 2\bar{x} = 2\lambda_1 A\bar{x} \quad A\bar{x} = \frac{1}{\lambda_1}\bar{x}$.

$\frac{1}{\lambda}$ is the eigenvalue associated with A.

$\bar{x}$ is the eigenvector

$Q(\vec{x}) = x^T A x = \frac{1}{\lambda}\|x\|^2 = 1$

$\therefore \|x\|^2 = \lambda$ therefore, if $\|x\|^2$ is minimized, λ is minimized, the eigenvalue $\frac{1}{\lambda}$ is maximized.