March. 22nd 2016 Homework # 6.

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1. a) strictly convex

b) concave

c) strictly concave

excellent!

2. $f(x,y) = -6x^2+(2a+4)xy-y^2-4ay$

$\frac{df}{dx} = -12x+(2a+4)y = -12x+2ay+4y$ $\quad$ $\frac{df}{dy} = (2a+4)x-2y-4a$
$= 2ax+4x-2y-4a$

$\frac{d^2f}{dx^2} = -12$ $\quad$ $\frac{d^2f}{dy^2} = -2$ $\quad$ $\frac{d^2f}{dxdy} = 2a+4$

For f to be convex, $\frac{d^2f}{dx^2}$ and $\frac{d^2f}{dy^2}$ must be greater than 0, and they are both less than 0 in this case. Therefore, for no value of a will make $f(x,y)$ a convex function.

4. $f(x,y,z) = x^a y^b z^c$ $\quad$ $\{(x,y,z) \mid x>0, y>0, z>0\}$

$f_x = ax^{a-1}(y^b z^c)$

$f_y = by^{b-1}(x^a z^c)$

$f_z = cz^{c-1}(x^a y^b)$

The Hessian matrix of $f(x,y,z)$:

$$= \begin{vmatrix} a(a-1)(x^{a-2})(y^b)(z^c) & ab(x^{a-1})(y^{b-1})(z^c) & ac(x^{a-1})(y^b)(z^{c-1}) \\ ab(x^{a-1})(y^{b-1})(z^c) & b(b-1)(x^a)(y^{b-2})(z^c) & bc(x^a)(y^{b-1})(z^c) \\ ac(x^{a-1})(y^b)(z^{c-1}) & bc(x^a)(y^{b-1})(z^{c-1}) & c(c-1)(x^a)(y^b)(z^{c-2}) \end{vmatrix}$$

$D_1 = a(a-1)(x^{a-2})(y^b)(z^c)$ ① $a(a-1)$ has to be less than 0 for f to be strictly concave.

$D_2 = a(a-1)(b)(b-1)(x^{2a-2})(y^{2b-2})(z^{2c}) - a^2b^2(x^{2a-2})(y^{2b-2})(z^{2c})$

$ab(a-1)(b-1) - a^2b^2 > 0$

$ab(ab-a-b+1) - a^2b^2 > 0$

$a^2b^2 - a^2b - ab^2 + ab - a^2b^2 > 0$

$ab - a^2b - ab^2 > 0$

② $ab(1-a-b) > 0$ for f to be strictly concave

D_3: $a(a-1)(b)(b-1)c(c-1) + (ab)(bc)(ac) + (ac)(ab)(bc)$
$- acb(b-1)ac - (bc)(bc)(a(a-1)) - (ab)(ab)(c(c-1))$
$= abc(a-1)(b-1)(c-1) + 2a^2b^2c^2 - a^2c^2b(b-1) - b^2c^2a(a-1)$
$- a^2b^2c(c-1)$
$= abc(a-1)(b-1)(c-1) + \cancel{2a^2b^2c^2} - a^2b^2c^2 + a^2c^2b - a^2b^2c^2 + ab^2c^2$
$- a^2b^2c^2 + a^2b^2c$
$= abc(a-1)(b-1)(c-1) + a^2c^2b + ab^2c^2 - a^2b^2c^2 + a^2b^2c$
$= abc(\cancel{abc} - \cancel{ac} - \cancel{bc} - \cancel{ab} + a + b + c - 1 + \cancel{ac} + \cancel{bc} + \cancel{ab} - \cancel{abc})$
$= abc(a+b+c-1) < 0$

③ $abc < 0$ or $a+b+c < 1$ for f to be strictly concave

Therefore if $a+b+c < 1$, it would satisfy all ①, ②, ③ making f strictly concave.

3. $g(x,y) = x - y - e^x - e^{x+y}$

$\frac{dg}{dx} = 1 - e^x - e^{x+y}$ $\frac{dg}{dy} = -1 - e^{x+y}$

$\frac{d^2g}{dx^2} = -e^x - e^{x+y}$ $\frac{d^2g}{dy^2} = -e^{x+y}$ $\frac{d^2g}{dxdy} = -e^{x+y}$

$\frac{d^2g}{dx^2} < 0$ $\frac{d^2g}{dy^2} < 0$ $\frac{d^2g}{dxdy} < 0$

$(-e^x - e^{x+y})(-e^{x+y}) - (-e^{x+y})^2$
$= (-e^{x+y})(-e^x - e^{x+y} + e^{x+y})$
$= (-e^{x+y})(-e^x)$
$= e^{2x+y} > 0$ $-e^x - e^{x+y} < 0$ strictly concave

5. $\left(\sqrt{\frac{x_1^2+\cdots x_n^2}{n}}\right)^2 = \frac{x_1^2+\cdots x_n^2}{n}$

$\left(\frac{x_1+\cdots x_n}{n}\right)^2 = \frac{(x_1+\cdots x_n)^2}{n^2}$

Consider $f = x^2$ is a convex function sincer $f''(x) = 2 > 0$

Jensen's inequality.

$\lambda_1 f(\vec{x}_1) + \cdots \lambda_n f(x_n) \geq f(\lambda_1 x_1 + \cdots \lambda_n x_n)$ if f is convex.

in this case $(\lambda_1 \ldots \lambda_n) = \frac{1}{n}$

$\frac{1}{n} f(x_1) + f(x_2) + \cdots \frac{1}{n} f(x_n) \geq f(\frac{1}{n}x_1 + \frac{1}{n}x_2 + \cdots \frac{1}{n}x_n)$

$\because \left(\sqrt{\frac{x_1^2+\cdots x_n^2}{n}}\right)^2 \geq \left(\frac{x_1+\cdots x_n}{n}\right)^2$

$\therefore \sqrt{\frac{x_1^2+\cdots x_n^2}{n}} \geq \frac{x_1+x_2+\cdots x_n}{n}$

x^{-1}

$-x^{-2}$

$\frac{1}{2}x^{-1}$

$-x^{-2}$

$2x^{-3}$

6. $f(x,y,z) = \frac{1}{xyz} + x + y + z \quad \{(x,y,z) \mid x>0, y>0, z>0\}$

$\frac{df}{dx} = \frac{1}{yz}\cdot\left(-\frac{1}{x^2}\right)+1 \qquad \frac{df}{dy} = \frac{1}{xz}\cdot\left(-\frac{1}{y^2}\right)+1 \qquad \frac{df}{dz} = \frac{1}{xy}\left(-\frac{1}{z^2}\right)+1$

$= -\frac{1}{x^2yz}+1 \qquad = -\frac{1}{xy^2z}+1 \qquad = -\frac{1}{xyz^2}+1$

$=0 \qquad =0 \qquad =0$

$$\begin{cases} \frac{1}{x^2yz} = 1 & x^2yz = 1 \\ \frac{1}{xy^2z} = 1 & xy^2z = 1 \\ \frac{1}{xyz^2} = 1 & xyz^2 = 1 \end{cases} \rightarrow (x,y,z) = (1,1,1)$$

Hessian matrix of $f(x,y,z)$:

$$\begin{vmatrix} \frac{2}{x^3yz} & \frac{1}{x^2y^2z} & \frac{1}{x^2yz^2} \\ \frac{1}{x^2y^2z} & \frac{2}{xy^3z} & \frac{1}{xy^2z^2} \\ \frac{1}{x^2yz^2} & \frac{1}{xy^2z^2} & \frac{2}{xyz^3} \end{vmatrix}$$

$D_1 = \frac{2}{x^3yz} > 0$

$D_2 = \frac{4}{x^4y^4z^2} - \frac{2}{x^4y^4z^2} > 0$

$$D_3 = \frac{8}{x^5y^5z^5} + \frac{1}{x^5y^5z^5} + \frac{1}{x^5y^5z^5} - \frac{2}{x^5y^5z^5} - \frac{2}{x^5y^5z^5} - \frac{2}{x^5y^5z^5}$$

$$= \frac{\cancel{2}4}{x^5y^5z^5} > 0$$

$\because D_1 > 0,\ D_2 > 0,\ D_3 > 0$

$\therefore f(x,y,z)$ is strictly convex over $\mathbb{R}^3$

$\therefore$ the point $(1,1,1)$ is a global minima.

$f(1,1,1) = 4$ is the global minima.

7. $f(x,y) = e^{x^2+y^4}$

e^x is strictly convex and increasing.

x^2+y^4 is convex

$e^{x^2+y^4}$ is convex.