# Homework 5 Selected Sol$^n$s

(3) The matrix of the given quadratic form is

$$A = \begin{pmatrix} -1 & 3 & 0 \\ 3 & -9 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

The columns of an orthogonal $P$ which makes $P^T A P$ diagonal are unit eigenvectors of $A$.

The char poly of $A$ is

$$\det \begin{pmatrix} -1-\lambda & 3 & 0 \\ 3 & -9-\lambda & 0 \\ 0 & 0 & -2-\lambda \end{pmatrix} = -(\lambda+2)(\lambda^2+10\lambda) = -(\lambda+2)\lambda(\lambda+10)$$

So the eigenvalues are $\lambda = -10, -2, 0$.

Eigenvector for $-10$:

$$\begin{pmatrix} 9 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 8 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \Rightarrow \begin{array}{c} 9x+3y=0 \\ 3x+y=0 \\ z=0 \end{array}$$

So a unit eigenvector is $\frac{1}{\sqrt{10}}(1,-3,0)$.

Eigenvector for $-2$:

$$\begin{pmatrix} 1 & 3 & 0 \\ 3 & 8 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

So a unit eigenvector is $(0,0,1)$

Eigenvector for $\lambda = 0$

$$\begin{pmatrix} -1 & 3 & 0 \\ 3 & -9 & 0 \\ 0 & 0 & -2 \end{pmatrix} = 0 \Rightarrow \begin{array}{c} -x+3y=0 \\ 3x-9y=0 \\ z=0 \end{array}$$

So a unit eigenvector is $\frac{1}{\sqrt{10}}(3,1,0)$

$$\Rightarrow P = \begin{pmatrix} 1/\sqrt{10} & 3/\sqrt{10} & 0 \\ -3/\sqrt{10} & 1/\sqrt{10} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(5) To check that $A = B^T B$ is symmetric:

$$\begin{aligned} A^T &= (B^T B)^T \\ &= B^T B^{TT} \\ &= B^T B \\ &= A \end{aligned}$$

Since $A^T = A$, $A$ is symmetric.

To check that $A$ is positive semidefinite, recall that the associated quadratic form is

$$Q(x) = x^T A x.$$

Now $x^T A x = x^T B^T B x$

$$= (Bx)^T Bx = \|Bx\|^2 \geq 0.$$

Furthermore, if $B$ is invertible, then $Bx \neq 0$ for $x \neq 0$, so $\|Bx\|^2 > 0$, so $Q$ is positive definite.

(6) Let $Q(x) = x^T A x$. There exists $P$ such that $P^T A P$ is diagonal. Let $x = Py$. Then

$$\begin{aligned} Q(x) &= y^T P^T A P y \\ &= y^T D y \\ &= \lambda_1 y_1^2 + \cdots + \lambda_n y_n^2 \end{aligned}$$

Since $Q$ is positive semidefinite, $\lambda_i \geq 0$. So

$$Q(x) = (\sqrt{\lambda_1}\, y_1)^2 + \cdots + (\sqrt{\lambda_n}\, y_n)^2$$

Now $y = P^{-1} x$, so the $y_i$ are just linear functions of the $x_1, \ldots, x_n$. Set $L_i(\bar{x}) = \sqrt{\lambda_i}\, y_i$. Then

$$Q(x) = L_1(\bar{x})^2 + \cdots + L_n(\bar{x})^2.$$