Excellent

(+0)

Analysis and Optimization : HW4

(1). (a). $\frac{\partial f}{\partial x}\Big|_{(2,1)} = y\Big|_{(2,1)} = 1 \qquad \frac{\partial f}{\partial y}\Big|_{(2,1)} = 2y + x\Big|_{(2,1)} = 4$

$\nabla f\Big|_{(2,1)} = (1, 4)$

(b) $\frac{\partial g}{\partial x}\Big|_{(0,0,1)} = e^{xy} + xye^{xy}\Big|_{(0,0,1)} = 1$

$\frac{\partial g}{\partial y}\Big|_{(0,0,1)} = x^2e^{xy}\Big|_{(0,0,1)} = 0 \qquad \frac{\partial g}{\partial z}\Big|_{(0,0,1)} = -2z\Big|_{(0,0,1)} = -2$

$\nabla g\Big|_{(0,0,1)} = (1, 0, -2)$

(c) $\frac{\partial h}{\partial x}\Big|_{(0,0,0)} = e^{x}\Big|_{(0,0,0)} = 1$

$\frac{\partial h}{\partial y}\Big|_{(0,0,0)} = 2e^{2y}\Big|_{(0,0,0)} = 2 \qquad \frac{\partial h}{\partial z}\Big|_{(0,0,0)} = 3e^{3z}\Big|_{(0,0,0)} = 3.$

$\nabla h\Big|_{(0,0,0)} = (1, 2, 3)$

(d) $\frac{\partial k}{\partial x}\Big|_{(0,0,0)} = e^{x+2y+3z}\Big|_{(0,0,0)} = 1$

$\frac{\partial k}{\partial y}\Big|_{(0,0,0)} = 2e^{x+2y+3z}\Big|_{(0,0,0)} = 2 \qquad \frac{\partial k}{\partial z}\Big|_{(0,0,0)} = 3e^{x+2y+3z}\Big|_{(0,0,0)} = 3$

$\nabla k\Big|_{(0,0,0)} = (1, 2, 3)$

(2). (a) let $\bar{n} = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

$\frac{\partial f}{\partial \bar{n}}(2,1) = \frac{\sqrt{2}}{2}\cdot\frac{\partial f}{\partial x}(2,1) + \frac{\sqrt{2}}{2}\frac{\partial f}{\partial y}(2,1) = \frac{\sqrt{2}}{2}(2+1) = \frac{3}{2}\sqrt{2}$

(b) let $\bar{n} = \left(\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right)$

$\frac{\partial f}{\partial \bar{n}}(0,1,1) = \frac{\sqrt{3}}{3}\frac{\partial f}{\partial x}(0,1,1) + \frac{\sqrt{3}}{3}\frac{\partial f}{\partial y}(0,1,1) + \frac{\sqrt{3}}{3}\frac{\partial f}{\partial z}(0,1,1)$

$= \frac{\sqrt{3}}{3}(0+0+(-2)) = -\frac{2}{3}\sqrt{3}$

(3) (a) $(-1,1,2)-(3,2,1)=(-4,-1,1)$

let $\vec{h}=\left(-\frac{4\sqrt{2}}{6},-\frac{\sqrt{2}}{6},\frac{\sqrt{2}}{6}\right)$

$$\frac{\partial f}{\partial \vec{h}}(1,1,1)=-\frac{4}{6}\sqrt{2}\cdot\frac{\partial f}{\partial x}\Big|_{(1,1,1)}-\frac{\sqrt{2}}{6}\frac{\partial f}{\partial y}\Big|_{(1,1,1)}+\frac{\sqrt{2}}{6}\frac{\partial f}{\partial z}\Big|_{(1,1,1)}.$$

$$=-\frac{4}{6}\sqrt{2}\left(\ln 3+\frac{2}{3}\right)-\frac{\sqrt{2}}{6}\left(\ln 3+\frac{2}{3}\right)+\frac{\sqrt{2}}{6}\cdot\frac{2}{3}$$

$$=-\frac{5\sqrt{2}}{6}\cdot\ln 3-\frac{4\sqrt{2}}{9}$$

(b) The direction of maximal increase is the direction of gradient.

$$\nabla f\Big|_{(1,1,1)}=\left(\ln 3+\frac{2}{3},\ln 3+\frac{2}{3},\frac{2}{3}\right)$$

(4) (a). $f(x,y)\approx f(0,0)+\frac{\partial f}{\partial x}\Big|_{(0,0)}x+\frac{\partial f}{\partial y}\Big|_{(0,0)}y+\frac{1}{2}\frac{\partial^2 f}{\partial x^2}\Big|_{(0,0)}x^2+\frac{1}{2}\frac{\partial^2 f}{\partial y^2}\Big|_{(0,0)}y^2+\frac{\partial^2 f}{\partial x\partial y}\Big|_{(0,0)}xy$

$=1+xy$

(b) $f(x,y)\approx 1+x^2-y^2$

(c) $f(x,y)\approx x+2y-\frac{1}{2}x^2-2y^2-2xy$

(5) $\nabla f=\left(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},\frac{\partial f}{\partial z}\right)=\left(3x^2+3y+3z,\ 3x+3y^2+3z,\ 3x+3y+3z^2\right)$

$$\nabla f=0 \iff \begin{cases}3x^2+3y+3z=0\\3x+3y^2+3z=0\\3x+3y+3z^2=0\end{cases} \iff \begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix} \text{ or } \begin{pmatrix}-2\\-2\\-2\end{pmatrix}$$

Stationary points: $(0,0,0)$, $(-2,-2,-2)$

(6). $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = \left(3x^2-3y, 3y^2-3x\right)$

$\nabla f = 0 \Rightarrow \begin{cases} 3x^2-3y=0 \\ 3y^2-3x=0 \end{cases} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$(0,0)$, $(1,1)$ are two stationary points

At $(0,0)$ $f(x,y) \approx -3xy$

At $(1,1)$ $f(x,y) \approx -1 + 3(x-1)^2 + 3(y-1)^2 - 3(x-1)(y-1)$

(7). degree 2 Taylor approximation of $\cos(x)$:

$\cos(x) \approx 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4$

$\Rightarrow (\cos(x))^2 \approx 1 - x^2 + \frac{1}{3}x^4$

(8). (a) $f'(0) = \lim_{x \to 0} \frac{e^{-\frac{1}{x^2}}}{x} \xrightarrow{\text{let } t=\frac{1}{x}} \lim_{t \to \infty} t e^{-t^2} = 0$

$f''(0) = \lim_{x \to 0} \frac{f'(x) - f'(0)}{x} = \frac{\frac{2}{x^3} e^{-\frac{1}{x^2}}}{x} \xrightarrow{\text{let } t=\frac{1}{x}} \lim_{t \to \infty} \frac{2t^4}{e^{t^2}} =$

(b) $\forall x \neq 0$. $f(x) = e^{-\frac{1}{x^2}} > 0$

$x=0$ $f(0) = 0$

$\Rightarrow f(0)$ is a local (global) minimum.