(1) (a) According to the graph given, $(0, \frac{10}{3})$, $(2,1)$, $(4,0)$ are the three vertices

$W_1 = 20 \times \frac{10}{3} = \frac{200}{3}$ $\quad W_2 = 14 \times 2 + 20 = 48$ $\quad W_3 = 4 \times 14 = 56$

$W_2 < W_3 < W_1$, so minimum: $W_2 = 48$.

(b) The dual problem: maximize $Z = 4y_1 + 20y_2$, subject to:

$y_1 + 7y_2 \le 14$, $\quad 2y_1 + 6y_2 \le 20$, $\quad 0 \le y_1$, $0 \le y_2$.

(c) The vertices are $(7,1)$, $(10,0)$ $(0,2)$, $(0,0)$

So the maximum: $Z = 4 \times 7 + 20 \times 1 = 48$.

(7,1)

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(2) First construct the minimization problem: minimize $W = 4x_1 + 2x_2$, subject to:

$x_1 + 2x_2 \ge 3$, $\quad 3x_1 + 2x_2 \ge 5$, $\quad 0 \le x_1, x_2$.

Then find the corresponding dual problem: maximize $Z = 3y_1 + 5y_2$, subject to:

$y_1 + 3y_2 \le 4$, $\quad 2y_1 + 2y_2 \le 2$, $\quad 0 \le y_1, y_2$.

Using the simplex method:

$$\begin{array}{c|ccccc} & y_1 & y_2 & s_1 & s_2 & b \\ \hline & 1 & 3 & 1 & 0 & 4 \\ & 2 & 2 & 0 & 1 & 2 \\ Z: & -3 & -5 & 0 & 0 & 0 \end{array} \longrightarrow \begin{array}{ccccc} y_1 & y_2 & s_1 & s_2 & b \\ \hline -2 & 0 & 1 & -\frac{3}{2} & 1 \\ 1 & 1 & 0 & \frac{1}{2} & 1 \\ 2 & 0 & 0 & \frac{5}{2} & 5 \end{array}$$

So $x_1 = s_1 = 0$, $x_2 = s_2 = \frac{5}{2}$. As a result, he buys no drink 1, $\frac{5}{2}$ liters of drink 2.

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(3) (a) $Z = 5x_2 - 3s_2 + 50$

$s_1 = x_2 - 2s_2 + 10$

$x_1 = -3s_2 + 20$

$s_3 = 3x_2 + 2s_2 + 40$

(b) $x_1 = 20$; $x_2 = 0$; $s_1 = 10$; $s_2 = 0$; $s_3 = 40$.

As we can observe from (a), the coefficients in front of x_2 are all positive numbers in deciding s_1, s_3, and z. So increasing x_2 will make s_1, s_2, z larger, with x_1 still equals 20. This incremement can be unbounded, so due to the linear relationship between x_2 and z, z can be made arbitrarily large in this way.

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(4) (a) an infeasibe problem: maximize $z = ax+by$, $\forall a,b \in \mathbb{R}$, subject to:

$x+y \le 4$, $x-y \ge 2$, $y \ge 2$.

(b) an unbounded problem: maximize $z = 2x+3y$, subject to:

$x+y \ge 4$, $x-y \ge 0$.

(c) Assume an unbounded linear programming problem's feasible set is bounded.

Then the set has finite vertices, feasible sets are closed $\Rightarrow$ it is compact

We know that optimal values are realized at vertices,

so now the problem's objective function attains its maximum & minimum on at least two of these vertices, according to maximum theorem.

$\Rightarrow$ this problem is bounded $\Rightarrow$ a contradiction $\square$.

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(5) (a) ① Maximize $c^T\bar{x}$, subject to: $A\bar{x} \le b$, $\bar{x} \ge 0$.

② Minimize $b^T\bar{y}$, subject to: $A^Ty \ge c$, $\bar{y} \ge 0$.

① and ② are dual problems. Denote the feasible sets for ①, ② by A, B, respectively.

Assume the dual minimization problem is feasible, $B \ne \phi$.

By week duality, $\forall \bar{x} \in A$, $\bar{y} \in B$, we have $c^T\bar{x} \le b^T\bar{y}$.

so cleary $c^T\bar{x}$ is bounded by $b^T\bar{y}$, for the smallest $\bar{y}$ in B

$\Rightarrow$ a contradiction $\square$.

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(b) Use the definition of ① and ② in (a).

Assume the dual maximization problem is feasible, then $A \ne \phi$.

By week duality, $\forall \bar{x} \in A$, $\bar{y} \in B$, $c^T\bar{x} \le b^T\bar{y}$.

So obviously $b^T\bar{y}$ is bounded by $c^T\bar{x}$, for the largest $\bar{x}$ in A

$\Rightarrow$ a contradiction $\square$.

4. (a) $Z = \sum_{i=1}^{n}\sum_{j=1}^{m} c_{ij} \cdot x_{ij}$

(b) $\sum_{j=1}^{m} x_{ij} \le a_i$;

$\sum_{i=1}^{n} x_{ij} \ge b_j$;

$x_{ij} \ge 0$, for all i, j.

(c) This problem can be modeled as follow: Given:

$$x_{11} + x_{12} \le a_1$$

$$x_{21} + x_{22} \le a_2$$

$$x_{31} + x_{32} \le a_3$$

$$x_{11} + x_{21} + x_{31} \ge b_1$$

$$x_{12} + x_{22} + x_{32} \ge b_2$$

minimize: $Z = 3x_{11} + x_{21} + 5x_{31} + 2x_{12} + 5x_{22} + 4x_{32}$

Calculated by an online program, the optimal solution is:

$Z = 200$. With $x_{11} = 0$, $x_{21} = 50$, $x_{31} = 0$, $x_{12} = 45$, $x_{22} = 0$, $x_{32} = 15$.

(d) (c) in standard form: minimize $Z = 3x_{11} + x_{21} + 5x_{31} + 2x_{12} + 5x_{22} + 4x_{32}$, subject to

$$-x_{11} - x_{12} \ge -a_1$$

$$-x_{21} - x_{22} \ge -a_2$$

$$-x_{31} - x_{32} \ge -a_3$$

$$x_{11} + x_{21} + x_{31} \ge b_1$$

$$x_{12} + x_{22} + x_{32} \ge b_2$$

$$x_{ij} \ge 0, \text{ for all } ij$$

So the dual problem can be written as: maximize $w = -45y_1 - 60y_2 - 35y_3 + 50y_4 + 60y_5$, subject to:

$$-y_1 + y_4 \le 3$$

$$-y_2 + y_4 \le 1$$

$$-y_3 + y_4 \le 2$$

$$-y_1 + y_5 \le 2$$

$$-y_2 + y_5 \le 5$$

The solution is : $w = 200$.

According to the weak duality, $w \le z$.

and its corollary, $w = z \Rightarrow z$ is an optimal solution □.

6)