# ANALYSIS AND OPTIMIZATION: HOMEWORK 3

SPRING 2016

**Due date: Wednesday, February 24.** In all the following, a "linear programming problem" refers to an optimization problem where we are trying to maximize or minimize a linear function subject to (closed) linear constraints.

(1) Problem 8 from LEF 9.4.

(2) Problem 22 from LEF 9.4.

(3) **When the simplex method breaks!**
Suppose we arrive at the following tableau in the simplex method for a maximization problem.

| | $x_1$ | $x_2$ | $s_1$ | $s_2$ | $s_3$ | $b$ | Basic |
|---|---|---|---|---|---|---|---|
| | 0 | $-1$ | 1 | 2 | 0 | 10 | $s_1$ |
| | 1 | 0 | 0 | 3 | 0 | 20 | $x_1$ |
| | 0 | $-3$ | 0 | $-2$ | 1 | 40 | $s_3$ |
| $z$ | 0 | $-5$ | 0 | 3 | 0 | 50 | |

(a) Write $z$ and the basic variables in terms of the non-basic variables.
(b) Write the values of $x_1, x_2, s_1, s_2, s_3$ corresponding to the tableau.
(c) Identify the entering variable. Note that our rule for picking a departing variable fails. Show that we can make $z$ arbitrarily large while satisfying the constraints $x_i \geq 0$ and $s_j \geq 0$.

**Note:** The above phenomenon is true in general. If all the entries in the entering variable column are non-positive, then the objective function can be made arbitrarily large on the feasible region.

(4) **Impossible problems**
A linear programming problem is called *infeasible* if the set of feasible solutions is empty. A maximization problem is called *unbounded* if the objective function can be made arbitrarily large on the feasible set. A minimization problem is called *unbounded* if the objective function can be made arbitrarily small on the feasible set. Give an example of an infeasible problem and an unbounded problem. Prove that if a linear programming problem is unbounded, then its feasible set must be unbounded.

(5) **Duality in impossibility**
Use weak duality to prove the following. If a maximization problem is unbounded, then the dual minimization problem is infeasible. Similarly, show that if a minimization problem is unbounded, then the dual maximization problem is infeasible.

(6) **A transportation problem** There are $n$ supply centers and $m$ demand centers. The $i$th supply center can supply at most $a_i$ units and the $j$th demand center needs at least $b_j$ units. Sending a unit from the $i$th supply center to the $j$th demand center costs $c_{ij}$. Denote by $x_{ij}$ the number of units sent from the $i$th warehouse to the $j$th outlet. Here $a_1, \ldots, a_n$, $b_1, \ldots, b_m$, and $c_{ij}$ are given constants and the $x_{ij}$ are the variables whose values we want to determine.

(a) Write down the objective function that we would like to minimize.

(b) Write down all the constraints.

(c) Consider the following specific problem

$$m = 2; n = 3; a_1 = 45; a_2 = 60; a_3 = 35; b_1 = 50; b_2 = 60,$$

and the cost matrix is

$$\begin{aligned} c_{11} &= 3, \quad c_{12} = 2 \\ c_{21} &= 1, \quad c_{22} = 5 \\ c_{31} &= 5, \quad c_{32} = 4. \end{aligned}$$

The problem might be too tedious to do by hand. Find a computer program on the internet that solves linear programs and use it to find the solution.

(d) Formulate the dual linear program. (Hint: First write the primal in a 'standard form': minimize $\gamma^T x$ subject to $\alpha x \geq \beta, x \geq 0$). Use the weak duality theorem to *prove* that the solution you found in the previous part is indeed optimal. In your proof, you will also have to use the solution to the dual linear program.

**Extra (not to be turned in):** Think about the interpretation of the shadow prices in the transportation problem.