Excellent!

HW 2

38/40

(1) Find the rank of the matrix.

$$\begin{bmatrix} 1 & 0 & -1 & 2 \\ -2 & 1 & 1 & -1 \\ -3 & 2 & 1 & 0 \end{bmatrix} \xrightarrow{2(1) \to (1)} \begin{bmatrix} 2 & 0 & -2 & 4 \\ -2 & 1 & 1 & -1 \\ -3 & 2 & 1 & 0 \end{bmatrix} \xrightarrow[(1)/2 \to (1)]{(2)-(1) \to (2)} \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & -1 & 3 \\ -3 & 2 & 1 & 0 \end{bmatrix} \xrightarrow{(1)3 \to (1)} \begin{bmatrix} 3 & 0 & -3 & 6 \\ 0 & 1 & -1 & 3 \\ -3 & 2 & 1 & 0 \end{bmatrix}$$

$(3)-(1) \to (3)$, $(1)/3 \to (1)$

$$\begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & -1 & 3 \\ 0 & 2 & -2 & 6 \end{bmatrix} \xrightarrow{(3)-2(2) \to (3)} \begin{bmatrix} \boxed{1} & 0 & -1 & 2 \\ 0 & \boxed{1} & -1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

RANK 2 : B/C 2 INDEPENDENT ROWS.

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(2)

$$\overbrace{\begin{bmatrix} A_1 & \cdots & A_n \\ \vdots & & \\ A_{1n} & & A_{nn} \end{bmatrix}}^{A} \overbrace{\begin{bmatrix} v_1 \\ \vdots \\ v_k \end{bmatrix}}^{b}$$

$\Rightarrow$ SINCE A IS INVERTIBLE, ITS COLUMNS ARE LINEARLY INDEPENDENT. $v_1 \dots v_k$ ARE ALSO LINEARLY INDEPENDENT SINCE THEY ARE NOT THE SAME VALUE OR A MULTIPLE OF EACH OTHER.

LET $w = Ab$ AND C BE A SCALAR. $c_1 w_1 + \dots c_n w_n = 0 \Rightarrow c_i = 0 \quad i = 1 \dots n$

SINCE A IS INVERTIBLE, MULTIPLYING THE LEFT SIDE BY A^{-1} SHOULD GIVE US $c_1 v_1 + \dots$

$\therefore$ SINCE $v_1 \dots v_k$ ARE LINEARLY IND. FOR ALL $c_i = 0$.

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(3)

$x - y + z = 2$

$x + 2y - z = 3$

$2x + y + 3z = 21$

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 2 & -1 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 21 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 1 & 2 & -1 & 3 \\ 2 & 1 & 3 & 21 \end{array}\right] \xrightarrow{(1)+(2)-(3) \to (1)} \left[\begin{array}{ccc|c} 0 & 0 & -3 & -16 \\ 1 & 2 & -1 & 3 \\ 2 & 1 & 3 & 21 \end{array}\right] \xrightarrow{(3)+(1) \to (3)} \left[\begin{array}{ccc|c} 0 & 0 & -3 & -16 \\ 1 & 2 & -1 & 3 \\ 2 & 1 & 0 & 5 \end{array}\right]$$

$(2)2 - (3)$

$$\begin{bmatrix} 0 & 0 & -3 \\ 0 & 3 & -2 \\ 2 & 1 & 0 \end{bmatrix}$$

$-3z = -16$

$\boxed{z = -5\,1/3}$

$3y - 2z = 1$

$3y - 2(5\,1/3) = 1$

$3y = \frac{35}{3}$

$\boxed{y = \frac{35}{9} = 3\frac{8}{9}}$

$2x + y = 5$

$2x + 3\,8/9 = 5$

$\boxed{x = 5/9}$

deg of freedom = 0, x, y, z determined by

3 degrees of freedom since each row is independent and is free to vary.

4/5

(4) Following set is closed in $\mathbb{R}^2$.

$$S = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \geq \sin(x+y),\ x \leq y,\ x^2+y^2 = 10xy\}$$

If prove all are closed, then intersection of set is closed.

$$S = \{(x,y) \mid \underbrace{x^2+y^2 - \sin(x+y) \geq 0}, \ \underbrace{x+y \geq 0}, \ \underbrace{x^2 - 10xy + y^2 = 0}\}$$

$$S = \{(x,y) \mid f(x,y) \geq 0,\ g(x,y) \geq 0,\ h(x,y) = 0\}$$

$f^{-1}(\{[0,+\infty)\})$ closed

$g^{-1}(\{[-\infty, 0]\})$ closed

$h^{-1}(\{[0]\})$ closed

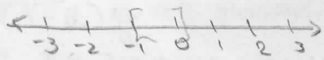
S is closed when these function intersect.

$x^2 - 10xy + y$

$x = 0$

$y = 0$

(5) $S \subset \mathbb{R}$ $\quad f: \mathbb{R} \to \mathbb{R}$



(a) S is closed but f(S) is not closed

$S = (-\infty, \infty)$

$f(x) = e^x$

$f(S) = (0, +\infty)$ ← does not contain boundary pt. not closed b/c it

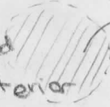
closed: contains all

(b) S is open but f(S) is not open.

$S = (0, 2\pi)$

$f(x) = \sin x$

$f(S) = [-1, 1]$ not open b/c two pts. are contained in f(S) but not interior pts.



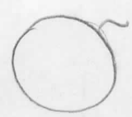
(c) S is bounded but f(S) is not bounded.

$S = (0, 1]$

$f(X) = 1/x$

$f(S) = (0, +\infty)$

not bounded b/c goes to infinity



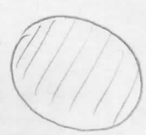
(d) S is compact but $f^{-1}(S)$ is not compact.

$S = [-\pi/2, \pi/2]$

$f(x) = \tan(x)$

$f^{-1}(S) = f^{-1}(\tan(x)) = \tan^{-1}(x)$

$f^{-1}(s) = (-\infty, \infty)$



the whole pict of shaded and boundary.

//// ∪ O

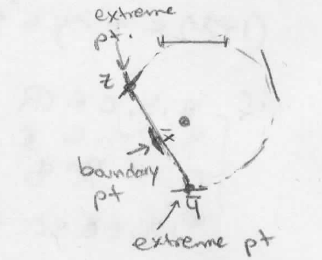
closed & bou

$(-\infty, -2]$ not b

$(2, 3)$ not clo

$[0, 1]$ closed

EXTREME PT. OF CONVEX SET IS A BOUNDARY PT. CONVERSE TRUE?



ASSUME S CONVEX SET $\rightarrow$ AN OPEN BALL $B_\Omega(x)$

IT HAS EXTREME POINTS SINCE THE TANGENT LINES ARE ~~NOT~~ INCLUDED AROUND THE BOUNDARY.

CONVERSE of it is $\boxed{\text{FALSE}}$

NOW:
SUPPOSE $x \in S$ AND x is AN INTERIOR PT.

WHERE $\exists\ B_\Omega(x) \in S$.

if $p, q \in S$ AND DRAW A DIAMETER ACROSS THROUGH x. $\overline{pq}$ SEGMENT MUST $\exists\ S$.

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good!

$\therefore$ Interior pt. cannot be an extreme pt, so it must be ~~converse~~ a boundary pt.

3/3

(7)

a. $(-\infty, 2] \cup \{5\} \rightarrow$ closed — interior pts: $(-\infty, 2)$ — boundary pts: $\{2, 5\}$

b. integers $\mathbb{Z}$ in $\mathbb{R} \rightarrow$ closed — no interior pts. $\{\emptyset\}$ — $\{\mathbb{Z}\}$

$\mathbb{Z}$ every single pt. boundary pt. b/c it part of complement part of s (neighborh

c. $\{\vec{x} \in \mathbb{R}^n \mid 4 \le |\vec{x}| \le 7\}$ in $\mathbb{R}^n \rightarrow$ closed — $\mathbb{R} \in$ btwn $|4|$ & $|7|$ — $|\vec{x}| = 4 \cup |\vec{x}| = 7$

$f^{-1}((4,7))$, $f(x) = \|x\|$

d. $\{(x,y) \in \mathbb{R}^2 \mid e^{x+y} \in (1,5)\}$ in $\mathbb{R}^2 \rightarrow$ open — $1 < e^{x+y} < 5$ — $\{e^{x+y} = 1\} \cup \{e^{x+y} = 5\}$

(x,y) s.t. this is true

$S = f^{-1}((1,5)) \rightarrow f(x) = e^{x+y}$ is continuous & open

e. $\{\frac{1}{n} \mid n = 1, 2, 3, 4, \ldots\}$ in $\mathbb{R} \rightarrow$ neither open or closed — ~~$(0,1)$~~ $\emptyset$ empty set — $\{0, 1\}$, $1/n\ \forall n$

never include 0 since $n \to \infty$.

4/8

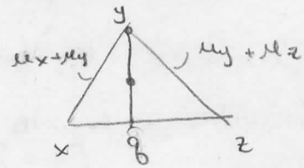
(8) S is convex set where $x, y \in S$

$(1-\lambda)x + \lambda y \in S \quad \lambda \in [0,1]$

if $a, b, c \in \mathbb{R}$ s.t. $a+b+c=1$

and

$x, y, z \in S$ s.t $ax+by+cz \in S$



Then: $a+b+c=1$ so; $1-c=a+b$

and

since $q \in \overline{xy}$ therefore $(1-\lambda)q + \lambda z \in S$

There is such α s.t. $(1-\alpha)q + \alpha z = ax+by+cz$

$\alpha z = cz \rightarrow \alpha = c$

$$(1-c)q + \cancel{cz} = ax+by+\cancel{cz}$$

$$q = \frac{ax+by}{1-c}$$

$\therefore \frac{a}{1-c}x + \frac{b}{1-c}y = q$. which is on the line segment $\overline{xy}$

To check=

$a+b+c=1$
$a+b=1-c$

$$\frac{a}{1-c} + \frac{b}{1-c} = \frac{a+b}{1-c} = \frac{1-c}{1-c} = 1 \checkmark$$

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