## ANALYSIS AND OPTIMIZATION: FINAL EXAM PRACTICE PROBLEMS

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### SOLUTIONS TO PRACTICE PROBLEMS

(1) Find all points $(x,y,z)$ that satisfy the first order conditions for local optima for $x+y+z$ subject to $x^2+y^2+z^2=1$ and $x-y-z=1$. Classify them as local maxima, local minima, or saddle points.

*Solution.* The Lagrange multiplier condition gives

$$1 = 2\lambda x + \mu$$
$$1 = 2\lambda y - \mu$$
$$1 = 2\lambda z - \mu.$$

The last two imply $2\lambda(y-z)=0$. So $\lambda=0$ or $y=z$. But $\lambda=0$ gives $1=\mu$ and $1=-\mu$, which is impossible. So $y=z$. The first two give $1=\lambda(x+y)$, so $\lambda=1/(x+y)$. Also, we have $x-y-z=x-2y=1$ and $x^2+y^2+z^2=x^2+2y^2=1$. Plugging in $x=1+2y$ in the last condition gives $6y^2+4y=0$, so $y=0$ or $y=-2/3$. Therefore, we get the following two points that satisfy the first order conditions: $(1,0,0)$ with $\lambda=1, \mu=-1$ and $(-1/3,-2/3,-2/3)$ with $\lambda=-1, \mu=1/3$.

The bordered Hessian is

$$B = \begin{pmatrix} 0 & 0 & 2x & 2y & 2z \\ 0 & 0 & 1 & -1 & -1 \\ 2x & 1 & -2\lambda & 0 & 0 \\ 2y & -1 & 0 & -2\lambda & 0 \\ 2z & -1 & 0 & 0 & -2\lambda \end{pmatrix}.$$

We need to check the size $m+r$ determinant for $m=2$ and $r=3$. For $(1,0,0)$ we get $(-1)^r \det B = 16$, which means this is a local maximum. For $(-1/3,-2/3,-2/3)$ we get $(-1)^m \det B = 8$, which means this is a local minimum. □

(2) Minimize the function $xy+x^2$ subject to $x^2+y \le 2$ and $y \ge 0$.

*Solution.* We want to maximize $-xy-x^2$ subject to $x^2+y\le 2$ and $-y \le 0$.

The KKT conditions are

$$-2x - y = 2\lambda x$$
$$-x = \lambda - \mu,$$

with $\lambda, \mu \ge 0$ and $\lambda = 0$ if $x^2+y<2$ and $\mu=0$ if $y>0$.

Suppose $y>0$ and $x^2+y<2$. Then $2x+y=0$ and $x=0$, so $(x,y)=(0,0)$.

Suppose $y>0$ and $x^2+y=2$. Then

$$-x=\lambda \implies -2x-y=-2x^2 \implies 2x^2-2x=2-x^2 \implies 3x^2-2x-2=0.$$

So $x = (1 \pm \sqrt{7})/3$. Since $\lambda = -x \geq 0$, we must discard $x = 1 + \sqrt{7}$ and keep $x = (1 - \sqrt{7})/3$. Then $y = (10 + 2\sqrt{7})/9$.

Suppose $y = 0$ and $x^2 + y < 2$. Then $x = \mu$ and $2x + y = 0$, so $x = 0$. So we get $(x, y) = (0, 0)$ again.

Finally, if $y = 0$ and $x^2 + y = 2$ then we get $(\pm\sqrt{2}, 0)$.

Let us check contraint qualification. The two gradients are $(2x, 1)$ and $(0, -1)$. CQ holds vacuously at points where neither constraint is binding. Since the gradient vectors are nonzero, CQ holds at the points where only one constraint is binding. The gradients are linearly independent at the two points that satisfy both constraints. So CQ holds at all points in the domain.

The domain is closed and bounded (the absolute value of $x$ and $y$ cannot be more than 2, for example). Hence a minimum is attained.

Checking the four points found above, we get that the minimum is attained at $x = (1 - \sqrt{7})/3, y = (10 + 2\sqrt{7})/9$. □

(3) Find the global minimum and maximum value of the function $xz$ on the set defined by $x + y - z = 0$, $x^2 + y \leq 2$, and $y \geq 1$.

*Solution.* Similar to the previous problem. □

(4) Consider the function $2x^2 + 4y^2$ on the set $x^2 + y^2 = 1$. Use Lagrange multipliers to find the global minimum and maximum of this function. What do the the second order criteria say at $(1, 0)$?

*Solution.* Global minimum is at $(\pm 1, 0)$. Global maximum is at $(0, \pm 1)$. The second order criterion says "local minimum" at $(2, 0)$. Details left to you. □

(5) Maximize $x + 2y$ subject to $3x^2 + y^2 \leq 1$, $x - y \leq 1$, $x \geq 0$, and $y \geq 0$.

(6) Let $c$ be a positive constant. Consider the problem of maximizing $\ln(x+1) + \ln(y+1)$ subject to $x + 2y \leq c$ and $x + y \leq 2$. Let $V(c)$ be the maximum value.
  (a) Find $V(5/2)$ and the $(x, y)$ that achieve this value.
  (b) Find $V'(5/2)$.

(7) A spring of natural length $L$ extended to length $L + x$ contains energy $\frac{1}{2}kx^2$, where $k$ is a constant called the *stiffness* of the spring. Suppose $n$ springs of natural lengths $L_1, \ldots, L_n$ and stiffnesses $k_1, \ldots, k_n$ are stringed together and the resulting contraption is extended to length $L_1 + \cdots + L_n + \ell$.
  (a) Find the extensions of the individual strings, assuming that the system minimizes the total energy. Justify why the solution you found is a global minimum.
  *Solution.* Suppose the total extension $\ell$ is produced by extensions $x_1, \ldots, x_n$ in the individual springs. Then $x_1 + \cdots + x_n = \ell$ and the total energy in the system is

$$\frac{1}{2}\left(k_1 x_1^2 + \cdots + k_n x_n^2\right).$$

  The Lagrange multiplier condition gives

$$k_i x_i = \lambda.$$

So $x_i = \lambda/k_i$. Since $\sum x_i = \ell$, we get

$$\lambda = \frac{\ell}{\sum 1/k_i},$$

and hence

$$x_1 = \frac{\ell}{k_1 \sum 1/k_i}, \ldots, x_n = \frac{\ell}{k_n \sum 1/k_i}.$$

To show that this is indeed a minimum, consider the Lagrangian

$$L = \frac{1}{2}\sum k_1 x_i^2 - \lambda(\sum x_i - \ell).$$

Its Hessian with respect to $x_1, \ldots, x_n$ is the matrix with $(k_1, \ldots, k_n)$ on the diagonal, which is positive definite. Therefore, the Lagrangian is a convex function of $x_1, \ldots, x_n$. Therefore, the point we found is a global minimum.

Note that the extension is inversely proportional to the stiffness, which makes sense. □

(b) Find the rate of change of the energy of the contraption with respect to $k_i$ and $L_i$ using the envelope theorem.

*Solution.* Denote by $E$ the energy of the contraption. The Lagrangian is

$$L = \frac{1}{2}\sum k_i x_i^2 - \lambda(\sum x_i - \ell).$$

Then

$$\frac{\partial E}{\partial k_i} = \frac{\partial L}{\partial k_j} = k_j x_j = \frac{\ell}{k_j \sum 1/k_i}.$$

There is no dependence on $L_i$. What I probably meant was the rate of change with respect to $\ell$, which Is

$$\frac{\partial E}{\partial \ell} = \lambda = \frac{\ell}{\sum 1/k_i}.$$

The last equation shows that the contraption behaves like a spring of stiffness

$$\frac{1}{\sum 1/k_i}.$$

□

(8) Use an appropriate bordered matrix to show that the quadratic form $-5x^2 + 2xy + 4xz - y^2 - 2z^2$ is negative definite on the subspace of $\mathbf{R}^3$ defined by $x + y + z = 0$ and $4x - 2y + z = 0$.

(9) Consider a closed box with sides $x, y, z$ and fixed volume $V$. Set up the Lagrange multiplier problem to minimize the surface area, find the candidate solution(s), and find the global minimum.

(10) Find the point on the line $x = y$ closest to the circle of radius 1 and center $(5, 2)$ using Lagrange multipliers. Make sure that your answer makes geometric sense. What is the

approximate change in the minimum distance if the center of the circle is moved from $(5,2)$ to $(5+\epsilon, 2-\epsilon)$?

*Solution.* We can write a point on the line $x = y$ as $(t,t)$ and a point on the circle as $(x,y)$ subject to $(x-5)^2+(y-2)^2 = 1$. We want to minimize $(x-t)^2+(y-t)^2$. The Lagrange multiplier condition gives

$$\begin{aligned}(x-t) &= \lambda(x-5)\\ (y-t) &= \lambda(y-2)\\ -2(x-t)-2(y-t) &= 0.\end{aligned}$$

The last condition combined with the first two gives $x-5 = 2-y$. With $(x-5)^2+(y-2)^2 = 1$, we get $x-5 = 2-y = \pm 1/\sqrt{2}$ and $t = (x+y)/2 = 7/2$. The Lagrangian

$$(x-t)^2+(y-t)^2-\lambda((x-5)^2+(y-2)^2-1)$$

will be convex if $\lambda < 0$, which is the case for $x-5 = -1/\sqrt{2}$. Therefore, the global minimum is achieved at

$$x = 5-1/\sqrt{2}, y = 2+1/\sqrt{2}, t = 7/2$$

with

$$\begin{aligned}\lambda &= \frac{5-1/\sqrt{2}-7/2}{-1/\sqrt{2}}\\ &= \frac{-(3-\sqrt{2})}{\sqrt{2}}.\end{aligned}$$

The minimum distance is $\frac{3\sqrt{2}-2}{2}$.

We can use the envelope theorem to find the rate of change. Let $D$ be the minimum distance from the circle centered at $(5+a, 2+b)$. Then (by the envelope theorem):

$$\begin{aligned}\frac{\partial D^2}{\partial a}|_{a=0} &= 2(x-5)\lambda,\\ \frac{\partial D^2}{\partial b}|_{b=0} &= 2(y-2)\lambda.\end{aligned}$$

So if $(a,b) = (\epsilon, -\epsilon)$, then

$$\begin{aligned}\frac{\partial D^2}{\partial \epsilon} &= (2(x-5)-2(y-2))\lambda\\ 2D\frac{\partial D}{\partial \epsilon} &= 2\sqrt{2}(3-\sqrt{2}).\end{aligned}$$

Therefore

$$\frac{\partial D}{\partial \epsilon} = \sqrt{2}.$$

□

*Much easier solution.* Minimizing the distance to the circle at (5,2) is equivalent to minimizing the distance to (5,2). So it suffices to minimize

$$(t-5)^2+(t-2)^2.$$

The minimum is achieved at

$$t = 7/2$$

and the minimum distance to $(5,2)$ is $3/\sqrt{2}$. Therefore the minimum distance to the circle is $3/\sqrt{2} - 1$.

Now let us take the circle to be at $(5 + a, 2 + b)$ as before. Let $D$ be the minimum distance to $(5,2)$. By the envelope theorem

$$\frac{\partial D^2}{\partial a}|_{a=0} = -2(t-5) = 3$$
$$\frac{\partial D^2}{\partial b}|_{b=0} = -2(t-2) = -3$$

Putting $(a, b) = (\epsilon, -\epsilon)$ we get

$$2D\frac{\partial D}{\partial \epsilon} = \frac{\partial D^2}{\partial \epsilon} = 6.$$

Therefore $\frac{\partial D}{\partial \epsilon} = 2/\sqrt{2} = \sqrt{2}$.

□

(11) Minimize

$$\int_0^1 x^2 + 2tx\dot{x} + \dot{x}^2 \, dt,$$

subject to $x(0) = 1$ and $x(1) = 2$. You may assume that a minimum exists.

*Solution.* The Euler–Lagrange equation is

$$2x + 2t\dot{x} - \frac{d}{dt}(2tx + 2\dot{x}) = 0$$
$$\ddot{x} = 0$$

So the genera solution is $x = at + b$. From the given conditions, we get $b = 1$ and $a = 1$. □

(12) The discounted total utility function for an investment strategy $K(t)$ over a period $T$ is given by

$$\int_0^T e^{-t/4} \ln(2K - \dot{K}) \, dt.$$

Find a function $K(t)$ that maximizes this subject to $K(0) = K_0$ and $K(T) = K_T$. You may assume that a maximum exists.

(13) By solving an Euler–Lagrange equation, find the curve of length $\pi$ joining $(0,0)$ and $(1,0)$ that together with the straight line from $(0,0)$ to $(1,0)$ encloses the maximum area.

*Hint:* This is a calculus of variations problem with a constraint.

*Proof.* When I solved this, I realized that a complete solution to this problem is more complicated than I had imagined. You should have been able to arrive at the differential equation to be solved, but it's OK if you did not solve it.

The problem is to maximize $\int_0^1 y\ dx$ subject to $\int_0^1 \sqrt{1+\dot{y}^2}\ dx = \pi$. In this case, the function $y - \lambda\sqrt{1+\dot{y}^2}$ satisfies the Euler–Lagrange equation for some $\lambda$. The equation is

$$1 - \lambda \frac{d}{dx}\left(\frac{\dot{y}}{\sqrt{1+\dot{y}^2}}\right) = 0.$$

Talk to me to see how to solve this equation, but you will not have to do something like this on the exam. $\square$