Solving a cubic equation

We want to explicitly find the roots of

$$X^3 + 3pX + 2q = 0.$$

Let u_1, u_2, u_3 be the roots. Key observation:

$$\mathbf{Q}(p, q, \omega) \subset \mathbf{Q}(p, q, \omega, \sqrt{\Delta}) \subset \mathbf{Q}(\omega, u_1, u_2, u_3).$$

We find eigenvectors for the A_3 action.

$$z = u_1 + \omega u_2 + \omega^2 u_3$$
$$z' = u_1 + \omega^2 u_2 + \omega u_3.$$

We know that

$$z^3, z'^3 \in \mathbf{Q}(\omega, p, q, \sqrt{\Delta}),$$

and

$$u_1 = \frac{z + z'}{3}, u_2 = \frac{\omega^2 z + \omega z'}{3}, u_3 = \frac{\omega z + \omega^2 z'}{3}.$$

Solving $X^3 + 3pX + 2q = 0$

$$
\begin{aligned}
z &= u_1 + \omega u_2 + \omega^2 u_3 \\
z^3 &= (u_1^3 + u_2^3 + u_3^3 + 6u_1u_2u_3) + \\
&\quad 3(u_1^2u_2 + u_2^2u_3 + u_3^2u_1)\omega + \\
&\quad 3(u_1u_2^2 + u_2u_3^2 + u_3u_1^2)\omega^2.
\end{aligned}
$$

$$
\begin{aligned}
u_1^3 + u_2^3 + u_3^3 + 6u_1u_2u_3 &= s_1^3 + 3s_1s_2 + 9s_3 = -18q \\
u_1^2u_2 + u_2^2u_3 + u_3^2u_1 &= A \\
u_1u_2^2 + u_2u_3^2 + u_3u_1^2 &= B
\end{aligned}
$$

Solving $X^3 + 3pX + 2q = 0$

$$A = u_1^2 u_2 + u_2^2 u_3 + u_3^2 u_1 \in \mathbf{Q}(\omega, p, q, \sqrt{\Delta})$$
$$B = u_1 u_2^2 + u_2 u_3^2 + u_3 u_1^2 \in \mathbf{Q}(\omega, p, q, \sqrt{\Delta})$$

Conjugates under $\mathrm{Gal}(\mathbf{Q}(\omega, p, q\sqrt{\Delta})/\mathbf{Q}(\omega, p, q))$. So,

$$A + B \in \mathbf{Q}(\omega, p, q)$$

and

$$(A - B)^2 \in \mathbf{Q}(\omega, p, q).$$

Solving $X^3 + 3pX + 2q = 0$

$$
\begin{aligned}
A + B &= u_1 u_2(u_1 + u_2) + u_2 u_3(u_2 + u_3) + u_3 u_1(u_3 + u_1) \\
&= s_1 s_2 - 3s_3 = 6q \\
A - B &= (u_1 - u_2)(u_2 - u_3)(u_1 - u_3) \\
&= \sqrt{\Delta} \\
\Delta &= -2^2 3^3 (q^2 + p^3).
\end{aligned}
$$

Solving $X^3 + 3pX + 2q = 0$: Plugging everything back

$$
\begin{aligned}
A &= \frac{(A+B)+(A-B)}{2} \\
&= 3q + \sqrt{-27(q^2+p^3)} \\
B &= 3q - \sqrt{-27(q^2+p^3)} \\
z^3 &= -18q + 3A\omega + 3B\omega^2 \\
&= -27q + 27\sqrt{q^2+p^3} \\
z'^3 &= -27q - 27\sqrt{q^2+p^3}
\end{aligned}
$$

Finally,

$$
\begin{aligned}
u_1 &= \frac{z+z'}{3} \\
&= \sqrt[3]{-q+\sqrt{q^2+p^3}} + \sqrt[3]{-q-\sqrt{q^2+p^3}}
\end{aligned}
$$