MODERN ALGEBRA 2: PRACTICE PROBLEMS FOR MIDTERM 2

The material on the midterm corresponds roughly to §12.3 – §12.5 and §15.1 – §15.5 of *Artin*. Almost all the problems in the corresponding exercises are very good, which you may use for practice. In addition, make sure you can do the homework problems and can prove the statements proved in class. Here is a selection of problems from *Artin* and elsewhere.

(1) Factor 10 as a product of Gaussian primes.

(2) Prove that two integer polynomials are relatively prime in $\mathbf{Q}[x]$ if and only if the ideal they generate in $\mathbf{Z}[x]$ contains a nonzero integer.

(3) Prove that the following polynomials are irreducible: x^2+1 in $\mathbf{F}_7[x]$ and x^3-9 in $\mathbf{F}_{31}[x]$.

(4) Factor x^5+5x+5 into irreducible factors in $\mathbf{F}_2[x]$ and $\mathbf{Z}[x]$.

(5) Determine the minimal polynomial of $\alpha = \sqrt{3}+\sqrt{5}$ over the following fields: $\mathbf{Q}$, $\mathbf{Q}(\sqrt{5})$, $\mathbf{Q}(\sqrt{3})$, $\mathbf{Q}(\sqrt{15})$.

(6) Let α and β be complex roots of two irreducible polynomials $f(x)$ and $g(x)$, respectively. Prove that $f(x)$ is irreducible over $\mathbf{Q}(\beta)$ if and only if $g(x)$ is irreducible over $\mathbf{Q}(\alpha)$. (*Hint:* Consider $\mathbf{Q}(\alpha,\beta)$.)

(7) Prove that x^4+3x+3 is irreducible over $\mathbf{Q}(\sqrt[3]{2})$. (*Hint:* Use the previous problem.)

(8) Let $F \subset L$ be a field extension. Let α and $\beta \in L$ have degree m and n over F, respectively. If $\gcd(m,n)=1$, show that the degree of $F(\alpha,\beta)$ over F is mn. Give a counterexample to show that the converse is not true.

(9) Let $F \subset L$ be a field extension. Let $K \subset L$ be the set of elements of L that are algebraic over F. Prove that K is a subfield of L.

(10) Let $F \subset L$ be a field extension. If $\alpha,\beta \in L$ are such that $\alpha+\beta$ and $\alpha\beta$ are algebraic over F, then show that both α and β are algebraic over F.

(11) Let $\zeta_n = e^{2\pi i/n}$. For which n does ζ_n have degree 2 over $\mathbf{Q}$?

(12) Let p be a prime number. What is the minimal polynomial of ζ_p over $\mathbf{Q}$?

(13) Let p be a prime number. Show that if a regular p-gon can be constructed by ruler and compass, then p must be of the form 2^n+1 for some n (*Hint:* Use the previous problem and that $\zeta_p = \cos(2\pi/p)+i\sin(2\pi/p)$.)

(14) Prove that a regular pentagon can be constructed by ruler and compass. (*Note:* You are not required to give an explicit construction.)

(15) Let $K=\mathbf{Q}(\alpha)$, where α is transcendental over $\mathbf{Q}$. Let β be an element of K that is not an element of $\mathbf{Q}$. Show that α is algebraic over $\mathbf{Q}(\beta)$.