MODERN ALGEBRA 2: PRACTICE PROBLEMS FOR MIDTERM 1

(1) Find (with proof) the kernel of the map $\mathbf{Q}[x] \to \mathbf{C}$ given by evaluation at $(i+2)$.

(2) Find (with proof) the kernel of the map $\mathbf{Z}[x] \to \mathbf{Z}_5$ given by $x \mapsto -1$.

(3) Determine the units in (1) $\mathbf{Z}/12\mathbf{Z}$, (2) $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/4\mathbf{Z}$, (3) $\mathbf{Z}[\sqrt{-2}]$.

(4) Find the gcd of x^2+x+1 and $x^4+3x^3+x^2+7x+5$ when considered as polynomials in $\mathbf{Q}[x]$ and when considered as polynomials in $\mathbf{F}_7[x]$.

(5) Give an example of a maximal ideal of $\mathbf{Z}[x]$. Give an example of a prime ideal of $\mathbf{Z}[x]$ which is not maximal. Give an example of an ideal of $\mathbf{Z}[x]$ which is not a prime ideal.

(6) Do the same problem for $\mathbf{F}[x,y]$, where $\mathbf{F}$ is your favorite field.

(7) Using $\mathbf{Z}/91\mathbf{Z} \cong \mathbf{Z}/7\mathbf{Z} \times \mathbf{Z}/13\mathbf{Z}$ compute $5^{26} \pmod{91}$.

(8) Describe the ring obtained from $\mathbf{Z}$ by adjoining α satisfying the two relations $2\alpha = 6$ and $6\alpha = 15$.

(9) Construct a finite field with 4 elements, 9 elements, 125 elements.

(10) Determine the maximal ideals of $\mathbf{C}[x]/x^2(x-1)$ and $\mathbf{R}[x]/(x^2-3x+2)$.

(11) Describe the ring $\mathbf{Z}[i]/(i+3)$. Is it a field?

(12) Let $\phi\colon R \to S$ be a ring homomorphism. Let I be an ideal of S. Show that $\phi^{-1}(I)$ is an ideal of R. If I is principal, must $\phi^{-1}(I)$ be principal? If I is maximal, must $\phi^{-1}(I)$ be maximal? If I is prime, must $\phi^{-1}(I)$ be prime? Prove or give counterexamples.

(13) The same questions assuming furthermore that ϕ is surjective.

(14) Let $a \in \mathbf{F}_p$. Factor $x^p - a$ in $\mathbf{F}_p[x]$.

(15) Is $\mathbf{Z}[\sqrt{-5}]$ a PID?

(16) Show that $(i+2)$ and $(i-2)$ are primes in $\mathbf{Z}[i]$.

(17) In general, show that if $a, b \in \mathbf{Z}$ and a^2+b^2 is prime, then $a+ib$ is prime in $\mathbf{Z}[i]$.