Modern Algebra 2: Midterm 1

Feb 20, 2014

Name: ____

- Write your answers in the space provided. Continue on the back for more space.
- Justify your answers unless asked otherwise.
- There are 6 questions.
- *Good luck*!

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
Total:	60	

1. (a) (5 points) Define an ideal.

Solution: An ideal of a ring R is a subset $I \subset R$ such that $0 \in I$, I is closed under addition, and if $i \in I$ and $r \in R$ then $ri \in R$.

(b) (5 points) Let $\phi \colon R \to S$ be a ring homomorphism. Prove that $\ker \phi$ is an ideal.

Solution: Let $I = \ker \phi$. Since $\phi(0) = 0$, we get $0 \in R$. If $a, b \in I$, then $\phi(a + b) = \phi(a) + \phi(b) = 0$ implies that $a + b \in I$. If $i \in I$ and $r \in R$, then $\phi(ri) = \phi(r)\phi(i) = 0$ implies that $ri \in I$. Thus, I is an ideal.

2. Determine all the ideals of the following rings:

(a) (5 points) $\mathbf{R}[x]/(x^2 - 3x + 2)$

Solution: By the correspondence theorem, the ideals of $\mathbf{R}[x]/(x^2 - 3x + 2)$ are in one to one correspondence with the ideals of $\mathbf{R}[x]$ that contain $(x^2 - 3x + 2)$. Since $\mathbf{R}$ is a field, the ideals of $\mathbf{R}[x]$ are all principal. An ideal $(p(x))$ contains $x^2 - 3x + 2$ if and only if $p(x)$ divides $x^2 - 3x + 2$. Since $x^2 - 3x + 2 = (x-1)(x-2)$, up to scalars, the only possibilities for $p(x)$ are 1, $(x-1)$, $(x-2)$, and $(x-1)(x-2)$. They correspond to the ideals (1), $(\overline{x} - 1)$, $(\overline{x} - 2)$ and (0) in $\mathbf{R}[x]/(x^2 - 3x + 2)$.

(b) (5 points) $\mathbf{R} \times \mathbf{R}$

Solution: There are many ways to do this; here is a slick way. By the Chinese remainder theorem,

$$\mathbf{R}[x]/(x^2 - 3x + 2) \cong \mathbf{R}[x]/(x-1) \times \mathbf{R}[x]/(x-2) \cong \mathbf{R} \times \mathbf{R}.$$

The isomorphism $\mathbf{R}[x]/(x^2 - 3x + 2) \cong \mathbf{R} \times \mathbf{R}$ is given by $p(x) \mapsto (p(1), p(2))$. Therefore, we can use the previous part to conclude that the ideals of $\mathbf{R} \times \mathbf{R}$ are generated by $(1,1)$, $(0,1)$, $(1,0)$ and $(0,0)$. More explicitly, these are $\mathbf{R} \times \mathbf{R}$, $\{0\} \times \mathbf{R}$, $\mathbf{R} \times \{0\}$ and $\{0\} \times \{0\}$.

3. (10 points) Let $\phi \colon R \to S$ be a ring homomorphism and $P \subset S$ a prime ideal. Show that $\phi^{-1}(P)$ is a prime ideal of R. Also give a counterexample to show that the statement is false if we replace 'prime' by 'maximal' everywhere.

Solution: Let $Q = \phi^{-1}(P)$. Take $a, b \in R$ such that $ab \in Q$. We must prove that $a \in Q$ or $b \in Q$. Since $ab \in Q$, we have $\phi(ab) = \phi(a)\phi(b) \in P$. Since P is prime, we must have $\phi(a) \in P$ or $\phi(b) \in P$, which means that $a \in Q$ or $b \in Q$.

For a counterexample for the same statement for 'maximal,' consider the inclusion $i \colon \mathbf{Z} \to \mathbf{Q}$. The ideal $\{0\} \subset \mathbf{Q}$ is maximal, but its preimage (which is also $\{0\}$) is not maximal.

4. Let $R = \mathbf{Z}[x]/(x^2 + x + 1)$.

(a) (5 points) Describe a ring homomorphism $\phi \colon R \to \mathbf{C}$.

Solution: A ring homomorphism $R \to \mathbf{C}$ corresponds to a ring homomorphism $\mathbf{Z}[x] \to \mathbf{C}$ under which $(x^2 + x + 1)$ is mapped to zero. A ring homomorphism $\mathbf{Z}[x] \to \mathbf{C}$ is specified completely by the image of x, which we call α. Under such a homomorphism $(x^2 + x + 1)$ is mapped to zero if and only if $\alpha^2 + \alpha + 1 = 0$. Therefore, we can choose $\alpha = \frac{-1+\sqrt{-3}}{2}$ or $\frac{-1-\sqrt{-3}}{2}$. Either choice gives us a homomorphism $R \to \mathbf{C}$.

(b) (5 points) Show that the equation $y^2 + 3 = 0$ has a solution in R.
Hint: You can use the previous part to find it.

Solution: Taking a hint from $x \mapsto \frac{-1+\sqrt{-3}}{2}$, we see that $2x + 1 \mapsto \sqrt{-3}$, and indeed $y = (2x + 1)$ satisfies $y^2 + 3 = 0$ in R. To check, we have

$$y^2 = (2x + 1)^2 = 4x^2 + 4x + 1 = 4(x^2 + x + 1) - 3 = -3 \text{ in } R.$$

5. Describe the following rings in as simple a way as you can:

(a) (5 points) $\mathbf{Z}[x]/(2x-6, 6x-15)$

Solution: Since $3 = (6x-15) - 3(2x-6)$, we have $3 \in (2x-6, 6x-15)$. Therefore,

$$\mathbf{Z}[x]/(2x-6, 6x-15) = \mathbf{Z}[x]/(3, 2x-6, 6x-15).$$

Going modulo 3 first, we get

$$\mathbf{Z}[x]/(3, 2x-6, 6x-15) = \mathbf{F}_3[x]/(2x).$$

Since 2 is a unit in $\mathbf{F}_3$, we have $(2x) = (x)$. Via the evaluation $x \mapsto 0$, we have $\mathbf{F}_3[x]/(x) \cong \mathbf{F}_3$. So, $\mathbf{Z}[x]/(2x-6, 6x-15) \cong \mathbf{F}_3$.

(b) (5 points) $\mathbf{Z}[\sqrt{3}]/(4+\sqrt{3})$

Solution: We have

$$\mathbf{Z}[\sqrt{3}]/(4+\sqrt{3}) \cong \mathbf{Z}[x]/(x^2+3, 4+x)$$
$$\cong \mathbf{Z}/(13) \text{ by } x \mapsto -4.$$

6. (10 points) Let $R = \mathbf{Z}[\sqrt{-2}]$. Show that 11 is not a prime element of R. Can you find a prime factor of 11? Be sure to justify that the factor you found is a prime.
Hint: It may be helpful to think of $\mathbf{Z}[\sqrt{-2}]$ *as* $\mathbf{Z}[x]/(x^2+2)$.

Solution: To show that 11 is not a prime, we try to show that $R/(11)$ is not a domain. We have

$$\begin{aligned} R/(11) &\cong \mathbf{Z}[x]/(11, x^2+2) \\ &\cong \mathbf{F}_{11}[x]/(x^2+2). \end{aligned}$$

But $(x^2+2) = (x-3)(x+3)$ in $\mathbf{F}_{11}[x]$. Therefore, $R/(11) \cong \mathbf{F}_{11}[x]/(x^2+2)$ is not a domain.

From the equation $(x-3)(x+3) = 0 \pmod{11}$ in R, we take a hint for factoring 11. Namely, we know that $(\sqrt{-2}-3)(\sqrt{-2}+3)$ is divisible by 11 in R, and indeed we get

$$11 = (3+\sqrt{-2})(3-\sqrt{-2}).$$

Let us show that one of these factors is a prime. Again, we compute $R/(3+\sqrt{-2})$. We have

$$\begin{aligned} R/(3+\sqrt{-2}) &\cong \mathbf{Z}[x]/(x^2+2, 3+x) \\ &\cong \mathbf{Z}/(11) \text{ via } x \mapsto -3. \end{aligned}$$

Since $R/(3+\sqrt{-2}) \cong \mathbf{Z}/(11)$ is a domain, we conclude that $3+\sqrt{-2}$ is prime.