MODERN ALGEBRA 2: HOMEWORK 6

In the last problem set, you showed the irreducibility of several specific polynomials. In this problem set, you will prove two general irreducibility criteria.

(1) Let $f(x) = a_n x^n + \cdots + a_0$ be a nonconstant polynomial in $\mathbf{Z}[x]$ and $p \in \mathbf{Z}$ a prime. Suppose p does not divide a_n and $f(x)$ is irreducible modulo p. Show that $f(x)$ is irreducible in $\mathbf{Q}[x]$. Exhibit a counterexample to show that the conclusion may fail if p divides a_n.

(2) (Eisenstein's criterion) Let $f(x) = a_n x^n + \cdots + a_0$ be a polynomial in $\mathbf{Z}[x]$ and $p \in \mathbf{Z}$ a prime. Suppose p does not divide a_n, p divides a_i for all $i < n$, and p^2 does not divide a_0. Show that $f(x)$ is irreducible $\mathbf{Q}[x]$.

(3) As an application, show that $1 + x + \cdots + x^{p-1}$ is irreducible in $\mathbf{Q}[x]$ (*Hint:* Replace x by $x+1$).

Let us exhibit a polynomial that is reducible modulo every prime p but irreducible in $\mathbf{Q}[x]$.

(4) Let $a, b \in \mathbf{F}_p$ be nonzero. Show that ab is a square in $\mathbf{F}_p$ if and only if either both a and b are squares or both a and b are non-squares.

(5) Consider

$$\begin{aligned} f(x) &= (x - \sqrt{2} - \sqrt{3})(x + \sqrt{2} + \sqrt{3})(x + \sqrt{2} - \sqrt{3})(x - \sqrt{2} + \sqrt{3}) \\ &= (x^2 - 1)^2 - 8x^2 \\ &= (x^2 + 1)^2 - 12x^2 \\ &= (x^2 - 5)^2 - 24. \end{aligned}$$

We know that $f(x)$ is irreducible in $\mathbf{Q}[x]$. Use the previous problem to show that $f(x)$ is reducible modulo every prime p.

Some problems from the book:

(6) Chapter 15, §2.3
(7) Chapter 15, §3.1
(8) Chapter 15, §3.3
(9) Chapter 15, §3.6
(10) Chapter 15, §3.10