MODERN ALGEBRA 1: HOMEWORK 8 (SOLUTION SKETCHES)

(1) Chapter 6: 7.1

Solution. The stabilizer of a vertex is $\{\mathrm{id}, s\}$ where s is the reflection in the line through the vertex and the origin.

The stabilizer of an edge is $\{\mathrm{id}, r\}$ where r is the reflection in the perpendicular bisector of the edge.

The stabilizer of a diagonal is $\{\mathrm{id}, r, s, \rho\}$ where r is the reflection in the diagonal, s the reflection in the other diagonal, and ρ the rotation by π. □

(2) Chapter 6: 7.2 (Take the line to be the X-axis for concreteness.)

Solution. The stabilizer of the X-axis is

$$\{t_v A \mid v \text{ parallel to the } X\text{-axis},\ A \in \{\mathrm{id}, r, s, \rho\}\},$$

where t_v is the translation by the vector v, r is the reflection in the X-axis, s is the reflection in the Y-axis, and ρ the rotation by π. Geometrically, the isometries $t_v A$ for the four possible A are horizontal translations, reflections in the X-axis followed by horizontal translations (glides), reflections in vertical lines, and rotations by π about points on the X-axis, respectively. □

(3) Chapter 6: 8.1

Solution. Yes, it does. Indeed,

$$I_n * A = I_n A I_n^t = A,$$

and

$$(MN) * A = MNA(MN)^t = MNAN^tM^t = M(NAN^t)M^t = M * (N * A).$$

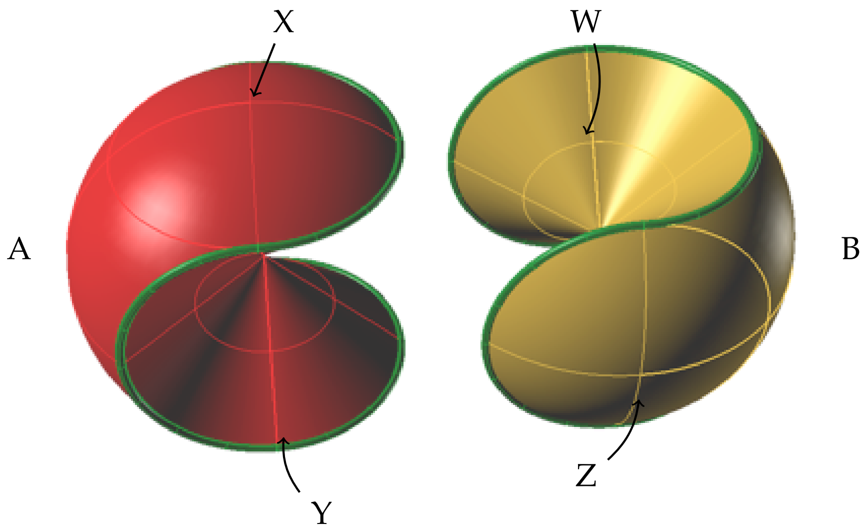
□

(4) Chapter 6: 9.3 (Feel free to look at a picture of a dodecahedron.)

Solution. Let G be the group of isometries of a dodecahedron. Consider the action of G on the set of faces. There are 12 faces and they form one orbit. Each face is a pentagon. The stabilizer of a face consists of the five rotations about the line joining the origin to the center of the face and the five reflections in the perpendicular bisector planes of the edges of the face. By the orbit stabilizer theorem, the order of G is $12 \cdot 10 = 120$. □

(5) Chapter 6: 9.6 (A tennis ball also has the same kind of seam. Feel free to work with an actual baseball or tennis ball. You will find some in the math help room.)

Solution. Let G be the group of isometries of the tennis ball. There are many ways to approach G, but the key is to find a convenient set of points whose orbit/stabilizers are tractable. It is helpful to visualize the ball as made up two pieces (red and yellow) joined at the seam (green).



Let A and B be the points on the surface of the ball at the center of the red and yellow areas, respectively. Then A and B form one orbit. For example, the reflection in the perpendicular bisector plane of AB followed by a rotation of $\pi/2$ about the line AB is an isometry of the ball that takes A to B. The stabilizer of A consists of the identity, the reflection in the horizontal plane through AB, the reflection in the vertical plane through AB and the rotation by π about AB. Thus, by orbit-stabilizer, we get $|G| = 2 \cdot 4 = 8$.

Identifying G is trickier. Consider X (the 'north pole' on the red piece), its antipode Y (the 'south pole' on the red piece), and Z, W (the two analogous points on the yellow piece.) In the picture, X and Z are at the intersection of the white cross-hairs, but W and Y are hidden from our view. Then $XZYW$ forms a square on the ball. It lies in the perpendicular bisector plane of AB. Note that every isometry of the ball gives an isometry of this square. We thus get a homomorphism $\phi\colon G \to D_4$. I'll leave it to you to check that ϕ is either injective or surjective. Since $|G| = |D_4|$, either implies that ϕ is an isomorphism. $\square$

(6) Let $\mathbf{R}^\times$ act on $\mathbf{R}^2$ by $t * (a,b) = (ta, t^{-1}b)$. Describe the orbits of this action. Which points have non-trivial stabilizers?

Solution. Suppose $a \neq 0$ and $b \neq 0$. Then the orbit of (a,b) is the hyperbola defined by $xy = ab$. The orbit of $(0,a)$ (for $a \neq 0$) is the Y-axis minus the origin. The orbit of $(a,0)$ (for $a \neq 0$) is the X-axis minus the origin. The orbit of $(0,0)$ is the singleton $\{(0,0)\}$.

Only $(0,0)$ has a non-trivial stabilizer, which is the entire $\mathbf{R}^\times$. $\square$