MODERN ALGEBRA 1: HOMEWORK 11

Problem. *Classify all groups of order up to 30 up to isomorphism.*

You may skip orders 8, 16, 24, and 27.

You may choose one of 12, 18, and 20, and skip the other two.

Concretely, the problem asks you to make a list of groups of order up to 30 such that every group of order up to 30 is isomorphic to a unique member of the list. The groups of order 8, 16, 24, and 27 require some ingenuity beyond what we have done in class. You are free to attempt to classify them, but you don't have to. The cases of orders 12, 18, and 20 are similar.

Make sure you prove that your list is exhaustive and does not contain repetitions. If you skip some orders, make it clear at the outset, so that the reader is not confused.

After having done the mathematics, think about the best way to write it up. One way is to simply go down the list numerically — order 1, order 2, order 3, order 4, etc — but you can save space and time, and help the reader by stating and proving some of the recurring arguments in general. For example, you can take care of all the primes at once.

You may find the following lemmas helpful. If you use them, you should include a proof. But you are not required to use them.

Lemma 1. *For every prime p, the group $\mathbf{Z}_p^\times$ is cyclic.*

If you cannot prove this in general, you can verify it directly for the few primes p that you need.

Lemma 2. *Let H and N be two groups, and consider two homomorphisms $\phi\colon H \to \operatorname{Aut} N$ and $\psi\colon H \to \operatorname{Aut} N$. If there is an automorphism of H, say $\alpha\colon H \to H$, such that $\phi = \psi \circ \alpha$, then the two semidirect products $N \rtimes_\phi H$ and $N \rtimes_\psi H$ are isomorphic.*

Lemma 3. *If N is a normal subgroup of G, and H is any subgroup of G, then the set NH defined by $NH = \{nh \mid n \in N, h \in H\}$ is a subgroup of G.*

The solution is due by Monday, December 9.