MODERN ALGEBRA 1: CLASSIFICATION OF GROUPS OF ORDER UP TO 30 (WITH SOME EXCEPTIONS)

We classify up to isomorphism the groups of order up to 30, excluding the orders 8, 16, 18, 24, and 27. We begin with a series of lemmas.

Lemma 1. *Let p be a prime. Then $\mathbf{Z}_p^\times$ is cyclic.*

Proof. The result is true in general, but the proof I know uses some tools beyond what we have done. We use it only for $p \leq 13$. It suffices to exhibit an element of $\mathbf{Z}_p^\times$ of order $(p-1)$. Here is a table for such elements for $p \leq 13$:

$$\begin{array}{c|cccccc} p & 2 & 3 & 5 & 7 & 11 & 13 \\ \hline & 1 & 2 & 2 & 3 & 2 & 2 \end{array}.$$

□

Lemma 2. *Let $\phi, \psi\colon H \to \operatorname{Aut} N$ be homomorphisms and $\alpha\colon H \to H$ an automorphism such that $\phi = \psi \circ \alpha$. Then we have an isomorphism $N \rtimes_\phi H \cong N \rtimes_\psi H$.*

Proof. Consider the function $f\colon N \rtimes_\phi H \to N \rtimes_\psi H$ given by

$$f(n,h) = (n, \alpha(h)).$$

We show that f is an isomorphism. Since α is a bijection, so is f. It remains to check that f is a homomorphism.

$$\begin{aligned} f((n_1,h_1)\cdot(n_2,h_2)) &= f(n_1\phi_{h_1}(n_2), h_1h_2) \\ &= (n_1\phi_{h_1}(n_2), \alpha(h_1h_2)). \end{aligned}$$

On the other hand,

$$\begin{aligned} f(n_1,h_1)\cdot f(n_2,h_2) &= (n_1,\alpha(h_1))\cdot(n_2,\alpha(h_2)) \\ &= (n_1\psi_{\alpha(h_1)}(n_2), \alpha(h_1)\alpha(h_2)) \\ &= (n_1\phi_{h_1}(n_2), \alpha(h_1h_2)). \end{aligned}$$

We thus get $f((n_1,h_1)\cdot(n_2,h_2)) = f(n_1,h_1)\cdot f(n_2,h_2)$. □

Lemma 3. *Let N and H be subgroups of G with $N \triangleleft G$. Then NH is a subgroup of G.*

Proof. We need to check that NH contains the identity, and is closed under products and inverses. Since $e = e\cdot e$, we get $e \in NH$. We have

$$n_1h_1 \cdot n_2h_2 = (n_1h_1n_2h_1^{-1})h_1h_2,$$

and since N is normal, $h_1n_2h_1^{-1} \in N$. Therefore, the product of n_1h_1 and n_2h_2 is of the form nh where $n = n_1h_1n_2h_1^{-1} \in N$ and $h = h_1h_2 \in H$. Similarly,

$$(n_1h_1)^{-1} = h_1^{-1}n_1^{-1} = h_1^{-1}n_1^{-1}h_1h_1^{-1},$$

which is of the form nh where $n = h_1^{-1}n_1^{-1}h_1 \in N$ and $h = h_1^{-1} \in H$. □

Lemma 4. *If N and H are subgroups of G such that $N \cap H = \{e\}$ and $|N||H| = G$, then $NH = G$. In particular, if $\gcd(|N|, |H|) = 1$ and $|N||H| = G$, then $NH = G$.*

Proof. Since $N \cap H = \{e\}$, we have $n_1 h_1 = n_2 h_2$ if and only if $n_1 = n_2$ and $h_1 = h_2$. That is, the products nh, for $n \in N$ and $h \in H$ are all distinct. Since there are $|N||H|$ of such products, and G has $|N||H|$ elements, we have $NH = G$. For the second statement, note that $N \cap H$ is a subgroup of both N and H. By Langrange's theorem, its order divides both $|N|$ and $|H|$. If $|N|$ and $|H|$ are coprime, then we must have $N \cap H = \{e\}$. □

We now have the tools to begin our classification.

Proposition 5. *Let G be a group of order p, then $G \cong \mathbf{Z}/p\mathbf{Z}$.*

Proof. In this case, G is the cyclic group generated by any non-identity element. □

Proposition 6. *Let G be a group of order p^2. Then $G \cong \mathbf{Z}_{p^2}$ or $G \cong \mathbf{Z}_p \times \mathbf{Z}_p$.*

Proof. We know that all groups of order p^2 are abelian. By the classification of abelian groups, the only two possibilities are those listed above. □

Proposition 7. *Let G be a group of order pq where $p < q$ are primes.*

(1) If p does not divide $q - 1$, then $G \cong \mathbf{Z}_p \times \mathbf{Z}_q$. Thus, there is only one group of order n up to isomorphism.
(2) If p divides $q - 1$, then either $G \cong \mathbf{Z}_p \times \mathbf{Z}_q$, or $G \cong \mathbf{Z}_p \rtimes_\phi \mathbf{Z}_q$, where $\phi \colon \mathbf{Z}_p \to \mathrm{Aut}(\mathbf{Z}_q)$ is any non-trivial homomorphism. Thus, there are two groups of order n up to isomorphism.

Proof. By the first Sylow theorem, G has a subgroup of order q. By the third Sylow theorem, the number of such subgroups divides p and is congruent to 1 modulo q. The only such number is 1, and hence there is a unique such subgroup. Call it N. Since the conjugates of N are also subgroups of order q, they must equal N. Therefore, N is a normal subgroup of G.

By the first Sylow theorem, G has a subgroup of order p. By Lemma 4, N and H are complementary. By Proposition 5, $N \cong \mathbf{Z}_q$ and $H \cong \mathbf{Z}_p$. Therefore,

$$G \cong \mathbf{Z}_q \rtimes_\phi \mathbf{Z}_p$$

for some $\phi \colon \mathbf{Z}_p \to \mathrm{Aut}\,\mathbf{Z}_q$. Recall that $\mathrm{Aut}\,\mathbf{Z}_q = \mathbf{Z}_q^\times$. By Lemma 1, we have an isomorphism $\mathbf{Z}_q^\times \cong \mathbf{Z}_{q-1}$. We now make two cases.

Case 1 (p does not divide $q - 1$): In this case, p and $q - 1$ are coprime, and hence the only homomorphism $\phi \colon \mathbf{Z}_p \to \mathrm{Aut}\,\mathbf{Z}_q \cong \mathbf{Z}_{q-1}$ is the trivial one. Therefore, $G \cong \mathbf{Z}_q \times \mathbf{Z}_p$.

Case 2 (p divides $q - 1$): In this case, let $(q - 1) = mp$. A homomorphism $\mathbf{Z}_p \to \mathbf{Z}_{q-1}$ must send 1 to an element x such that $px \equiv 0 \pmod{(q-1)}$. Thus, we get p different homomorphism ϕ_i given by $\phi_i \colon 1 \mapsto im$ for $i = 0, \ldots, (p - 1)$. Then $G \cong \mathbf{Z}_q \rtimes_{\phi_i} \mathbf{Z}_p$ for some ϕ_i. However, for every $1 \le i \le p - 1$, we have an automorphism $\alpha_i \colon \mathbf{Z}_p \to \mathbf{Z}_p$ given by multiplication by i, and $\phi_i = \phi_1 \circ \alpha_i$. Therefore, by Lemma 2, the semidirect products correspding to $\phi_1, \ldots, \phi_{p-1}$ are in fact isomorphic. We thus get $G \cong \mathbf{Z}_q \times \mathbf{Z}_p$ (corresponding to ϕ_0) or $G \cong \mathbf{Z}_q \rtimes_\phi \mathbf{Z}_p$ (corresponding to any non-trivial ϕ). Since $\mathbf{Z}_q \times \mathbf{Z}_p$ is abelian, while $\mathbf{Z}_q \rtimes_\phi \mathbf{Z}_p$ is not, they are not isomorphic.

□

Proposition 5, Proposition 6, and Proposition 7 cover the orders 2, 3, 4, 5, 6, 7, 9, 10, 11, 13, 14, 15, 17, 19, 21, 22, 23, 25, 26, and 29. We are thus left with orders 8, 12, 16, 18, 20, 24, 27, 28, and 30.

Proposition 8. *There are five groups of order 12 up to isomorphism:* $\mathbf{Z}_{12}$, $\mathbf{Z}_2 \times \mathbf{Z}_6$, $(\mathbf{Z}_2 \times \mathbf{Z}_2) \rtimes \mathbf{Z}_3$, $\mathbf{Z}_3 \rtimes \mathbf{Z}_4$, *and* $\mathbf{Z}_3 \rtimes (\mathbf{Z}_2 \times \mathbf{Z}_2)$.

Proof. Let G be a group of order 12. We first show that G has a normal subgroup of order 3 or 4. By the third Sylow theorem, G has either one or four 3-Sylow subgroups. Suppose the 3-Sylow subgroup is not normal. Then there are four 3-Sylow subgroups. Since any non-identity element of a 3-element group generates the group, two distinct 3-Sylow subgroups cannot share a non-identity element. We thus get $4 \cdot 2 = 8$ elements in G of order 3. A 2-Sylow subgroup of G has order 4 and it cannot contain any of the 8 elements of order 3. It follows that there can be only one 2-Sylow subgroup. In particular, it must be normal.

We now make cases.

Case 1 (2-Sylow is normal): Let N be a 2-Sylow subgroup. By Lemma 4 a 3-Sylow subgroup H is a complementary group, and hence $G \cong N \rtimes H$. We have $H \cong \mathbf{Z}_3$. For N, we make two subcases.

Case 1 (a) ($N \cong \mathbf{Z}_4$): Then $G \cong \mathbf{Z}_4 \rtimes_\phi \mathbf{Z}_3$. Since $\operatorname{Aut} \mathbf{Z}_4 = \mathbf{Z}_4^\times \cong \mathbf{Z}_2$ and there are no nontrivial automorphisms from $\mathbf{Z}_3 \to \mathbf{Z}_2$, we have $G \cong \mathbf{Z}_4 \times \mathbf{Z}_3 \cong \mathbf{Z}_{12}$.

Case 1 (b) ($N \cong \mathbf{Z}_2 \times \mathbf{Z}_2$): Then $G \cong \mathbf{Z}_4 \rtimes_\phi \mathbf{Z}_3$ for some $\phi\colon \mathbf{Z}_3 \to \operatorname{Aut}(\mathbf{Z}_2 \times \mathbf{Z}_2)$. The homomorphism ϕ is determined by $\phi(1)$. An automorphism of $\mathbf{Z}_2 \times \mathbf{Z}_2$ must fix $(0,0)$ and permute the three other elements. Furthermore, $\phi(1)$ must satisfy $\phi(1)^3 = \mathrm{id}$. Therefore, either $\phi(1) = \mathrm{id}$ (in which case ϕ is trivial) or $\phi(1)$ acts as a 3-cycle on the three non-identity elements. It is easy to check that both such 3-cycles do give automorphisms of $\mathbf{Z}_2 \times \mathbf{Z}_2$. Since there are only two such 3-cycles, we have at most two possibilities for ϕ, say ϕ_1 and ϕ_2. However, if we denote by $\alpha\colon \mathbf{Z}_3 \to \mathbf{Z}_3$ the automorphism $x \mapsto -x$, then $\phi_1 \circ \alpha = \phi_2$. Therefore, by Lemma 2, they give isomorphic semidirect products. We thus get $G \cong \mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_3$ or $(\mathbf{Z}_2 \times \mathbf{Z}_2) \rtimes \mathbf{Z}_3$ (nontrivial).

Case 2 (3-Sylow is normal): Let N be the 3-Sylow subgroup. Then $N \cong \mathbf{Z}_3$. Let H be a 2-Sylow subgroup. By Lemma 4, N and H are complementary, and hence $G \cong \mathbf{Z}_3 \rtimes H$. Again we have two cases.

Case 2 (a) ($H \cong \mathbf{Z}_4$): Then $G \cong \mathbf{Z}_3 \rtimes_\phi \mathbf{Z}_4$. There are two homomorphisms $\phi\colon \mathbf{Z}_4 \to \operatorname{Aut} \mathbf{Z}_3 \cong \mathbf{Z}_2$, one trivial and one non-trivial. This gives two semidirect products $\mathbf{Z}_3 \times \mathbf{Z}_4$ and $\mathbf{Z}_3 \rtimes \mathbf{Z}_4$ (nontrivial).

Case 2 (b) ($H \cong \mathbf{Z}_2 \times \mathbf{Z}_2$): Then $G \cong \mathbf{Z}_3 \rtimes (\mathbf{Z}_2 \times \mathbf{Z}_2)$. There are four homomorphisms $\mathbf{Z}_2 \times \mathbf{Z}_2 \to \operatorname{Aut} \mathbf{Z}_3 \cong \mathbf{Z}_2$, one trivial and three non-trivial. The three non-trivial ones are related to each other by composing with automorphisms of $\mathbf{Z}_2 \times \mathbf{Z}_2$. Therefore, by Lemma 2, the semidirect products they give are isomorphic to each other. We thus get two semidirect products $\mathbf{Z}_3 \times \mathbf{Z}_2 \times \mathbf{Z}_2$ and $\mathbf{Z}_3 \rtimes (\mathbf{Z}_2 \times \mathbf{Z}_2)$ (nontrivial).

□

Remark 9. We may choose a more friendly list of 5 groups of order 12: $\mathbf{Z}_{12}$, $\mathbf{Z}_2 \times \mathbf{Z}_6$, D_6, $\mathbf{S}_3 \times \mathbf{Z}_2$, and $\mathbf{Z}_3 \rtimes \mathbf{Z}_4$. Let us check that no two of these are isomorphic. The first two are abelian while the last three are not. The first is cyclic while the second is not. The third has 6 elements of order 2 while the fourth has 4. The last one has $\mathbf{Z}_4$ as its 2-Sylow subgroup while the third and the fourth have $\mathbf{Z}_2 \times \mathbf{Z}_2$. Combining these observations, we see that no two are isomorphic.

Proposition 10. *There are five groups of order 20 up to isomorphism:* $\mathbf{Z}_{20}$, $\mathbf{Z}_2 \times \mathbf{Z}_{10}$, $\mathbf{Z}_5 \rtimes_1 \mathbf{Z}_4$, $\mathbf{Z}_5 \rtimes_2 \mathbf{Z}_4$, *and* $\mathbf{Z}_5 \rtimes (\mathbf{Z}_2 \times \mathbf{Z}_2)$.

Proof. Let G be a group of order 20. The Sylow theorems imply that G has a normal subgroup N of order 5 and a complementary subgroup H of order 4. Then $G \cong N \rtimes H$. There are two cases.

Case 1 ($H \cong \mathbf{Z}_4$): Then $G \cong \mathbf{Z}_5 \rtimes_\phi \mathbf{Z}_4$. There are four homomorphisms $\mathbf{Z}_4 \to \operatorname{Aut} \mathbf{Z}_5 \cong \mathbf{Z}_4$ corresponding to $1 \mapsto 0, 1, 2$, or 3. The two, $1 \mapsto 1$ and $1 \mapsto 3$, are related by an automorphism α of $\mathbf{Z}_4$ given by $\alpha \colon x \to -x$. Therefore, we get three semidirect products $\mathbf{Z}_5 \times \mathbf{Z}_4$, $\mathbf{Z}_5 \rtimes_1 \mathbf{Z}_4$, and $\mathbf{Z}_5 \rtimes_2 \mathbf{Z}_4$. The last two are indeed non-isomorphic. The center of $\mathbf{Z}_5 \rtimes_1 \mathbf{Z}_4$ is trivial, whereas the center of $\mathbf{Z}_5 \rtimes_2 \mathbf{Z}_4$ has a non-identity element, namely the $2 \in \mathbf{Z}_4$.

Case 2 ($H \cong \mathbf{Z}_2 \times \mathbf{Z}_2$): Then $G \cong \mathbf{Z}_5 \rtimes_\phi \mathbf{Z}_2 \times \mathbf{Z}_2$. There are four homomorphisms $\mathbf{Z}_2 \times \mathbf{Z}_2 \to \operatorname{Aut} \mathbf{Z}_5 \cong \mathbf{Z}_4$, one trivial and three non-trivial. The three non-trivial ones are related to each other by automorphisms of $\mathbf{Z}_2 \times \mathbf{Z}_2$. Hence, in this case, we get two semidirect products $\mathbf{Z}_5 \times \mathbf{Z}_2 \times \mathbf{Z}_2$ and $\mathbf{Z}_5 \rtimes (\mathbf{Z}_2 \times \mathbf{Z}_2)$ (nontrivial).

□

Remark 11. A more friendly list is: $\mathbf{Z}_{20}$, $\mathbf{Z}_2 \times \mathbf{Z}_{10}$, D_{10}, and two non-trivial semidirect products $\mathbf{Z}_5 \rtimes_1 \mathbf{Z}_4$ and $\mathbf{Z}_5 \rtimes_2 \mathbf{Z}_4$.

Proposition 12. *There are four groups of order 28 up to isomorphism:* $\mathbf{Z}_{28}$, $\mathbf{Z}_2 \times \mathbf{Z}_{14}$, $\mathbf{Z}_7 \rtimes \mathbf{Z}_4$, *and* $\mathbf{Z}_7 \rtimes (\mathbf{Z}_2 \times \mathbf{Z}_2)$.

Proof. The analysis here is similar to the case of order 20, but easier. Let G be a group of order 20. The Sylow theorems imply that G has a normal subgroup N of order 7 and a complementary subgroup H of order 4. Then $G \cong N \rtimes H$. There are two cases.

Case 1 ($H \cong \mathbf{Z}_4$): Then $G \cong \mathbf{Z}_7 \rtimes_\phi \mathbf{Z}_4$. There are two homomorphisms $\mathbf{Z}_4 \to \operatorname{Aut} \mathbf{Z}_7 \cong \mathbf{Z}_6$ corresponding to $1 \mapsto 0$ or 3. Therefore, we get two semidirect products $\mathbf{Z}_7 \times \mathbf{Z}_4$ and $\mathbf{Z}_7 \rtimes \mathbf{Z}_4$ (nontrivial).

Case 2 ($H \cong \mathbf{Z}_2 \times \mathbf{Z}_2$): Then $G \cong \mathbf{Z}_7 \rtimes_\phi \mathbf{Z}_2 \times \mathbf{Z}_2$. There are four homomorphisms $\mathbf{Z}_2 \times \mathbf{Z}_2 \to \operatorname{Aut} \mathbf{Z}_7 \cong \mathbf{Z}_6$, one trivial and three non-trivial. The three non-trivial ones are related to each other by automorphisms of $\mathbf{Z}_2 \times \mathbf{Z}_2$. Hence, in this case, we get two semidirect products $\mathbf{Z}_7 \times \mathbf{Z}_2 \times \mathbf{Z}_2$ and $\mathbf{Z}_7 \rtimes (\mathbf{Z}_2 \times \mathbf{Z}_2)$ (nontrivial).

□

Remark 13. A more friendly list is: $\mathbf{Z}_{28}$, $\mathbf{Z}_2 \times \mathbf{Z}_{14}$, D_{14}, and $\mathbf{Z}_7 \rtimes \mathbf{Z}_4$.

Proposition 14. *There are four groups of order 30 up to isomorphism; all four are semidirect products* $\mathbf{Z}_{15} \rtimes_\phi \mathbf{Z}_2$.

Proof. Let G be of order 30. We claim that G has a normal subgroup of order 3 or order 5. Note that the number of 3-Sylow subgroups is either 1 or 10 and the number of 5-Sylow subgroups is either 1 or 6. If the 3-Sylow is not normal, then we get $10 \cdot 2 = 20$ elements in G of order 3. If the 5-Sylow is not normal, then we get $4 \cdot 6 = 24$ elements in G of order 5. Clearly, both possibilities cannot occur at the same time.

Suppose the 3-Sylow subgroup N is normal. Let H be a 5-Sylow subgroup. Then $N' = NH$ is a subgroup of order 15. Likewise, if the 5-Sylow subgroup N is normal. Let H be a 3-Sylow subgroup. Then $N' = NH$ is a subgroup of order 15. In either case, we get a subgroup of G of order 15, which must be normal, since it has index two. Let H' be a 2-Sylow subgroup. Then H' is complementary to N'. By the classification of groups of order 15, we get $N' \cong \mathbf{Z}_{15}$. We know that $H' \cong \mathbf{Z}_2$. Thus $G \cong \mathbf{Z}_{15} \rtimes_\phi \mathbf{Z}_2$. There are four homomorphisms $\phi \colon \mathbf{Z}_2 \to \mathrm{Aut}(\mathbf{Z}_{15}) = \mathbf{Z}_{15}^\times$ corresponding to $1 \mapsto 1, 4, 11$, or 14. We thus get at most four groups of order 30 up to isomorphism.

On the other hand, $\mathbf{Z}_{30}$, D_{15}, $\mathbf{S}_3 \times \mathbf{Z}_5$, and $D_5 \times \mathbf{Z}_3$ are four pairwise non-isomorphic groups of order 30. To see that they are pairwise non-isomorphic, we can count the number of elements of order 2 in each one; these numbers are 1, 15, 3, and 5, respectively. Hence these are the only four groups of order 30, up to isomorphism. $\square$

Remark 15. A more friendly list is: $\mathbf{Z}_{30}$, D_{15}, $\mathbf{S}_3 \times \mathbf{Z}_5$, and $D_5 \times \mathbf{Z}_3$.