Mathematics W4041x
Introduction to Modern Algebra

Practice Final Exam

December 16, 2010

1. State the classification of finitely generated abelian groups.

2. If $\#G = 20$ and $S \subset G$ with $\#S = 12$, must $\langle S \rangle = G$? Why or why not?

3. What are the possible numbers of Sylow 3-subgroups in a group of order 210?

4. Let S be a set, PS its power set. For $A, B \in PS$, say $A \sim B$ if there exists a bijection $A \to B$. Prove that $\sim$ is an equivalence relation.

5. Can a group of order p^n, where p is prime and $n > 1$, ever be simple? Why or why not?

6. Classify the groups of order 21 up to isomorphism. How many are there?

7. For each prime p dividing $\#\Sigma_4$, describe the Sylow p-subgroup of Σ_4 in terms of familiar groups.

8. Prove that the quaternion group Q is *not* isomorphic to a semidirect product except in a trivial fashion as $Q \rtimes 1$ or $1 \rtimes Q$.

9. If G and H are finite simple groups and $K \lhd G \times H$, prove that K is isomorphic to 1, G, H, or $G \times H$.

10. Prove that if $\sigma, \tau \in \Sigma_n$, then $\sigma\tau$ and $\tau\sigma$ factor into disjoint cycles of the same sizes.

11. (a) If $N \lhd G$, prove that conjugation defines an action of G on N by automorphisms.

(b) If $N \lhd G$, $\#N = 5$, and $\#G$ is odd, prove that $N \subset ZG$, the center of G.

12. Prove that a finite abelian group whose order is not divisible by the square of any prime must be cyclic.

13. If G is a finite group with $H < G$ and $N \lhd G$, and if $[G : N]$ and $\#H$ are relatively prime, prove that $H < N$.

14. A *pentagonal prism* is the set of $(x, y, z) \in \mathbb{R}^3$ such that (x, y) lies in a regular pentagon and $z \in [0, 1]$, as sketched below.

(a) Describe, without proof, the group of rotations of this prism.

(b) How many inequivalent ways are there to paint the 7 faces of this prism with 3 colors (blue, red, purple)?