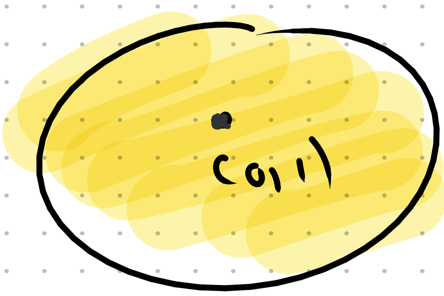
Exercise about proof of completeness

$$Z = V(X^2 - sY^2,\ sX + tY) \subset \mathbb{P}^1 \times \mathbb{A}^2$$

$$\downarrow$$

We proved $\mathrm{Im}(Z) \subset \mathbb{A}^2$ is closed.

How? Take $(0,1) \notin \mathrm{Im}\, Z$



$$X^2 - 0\cdot Y^2 = 0 \ ; \quad 0\cdot X + 1\cdot Y = 0$$

$\exists n : \langle X^2, Y\rangle \supset \langle X, Y\rangle^n$

$\boxed{\text{Pick}}$ $\langle X^2, Y\rangle \supset \langle X^3, X^2Y, XY^2, Y^3\rangle$

6

$(s,t) = (0,1)$

$n=3 \mid n=2$

$n=4$

$X^2 \cdot (\text{Linears}) + Y \cdot (\text{quadratics})$

$\Rightarrow (X^2 - sY^2) \cdot (\underline{\text{Linear}}) +$
$(sX + tY) \cdot (\underline{\text{Quadratic}})$

All cubics are of this form for all (s,t) in some n'hood of $(0,1)$. ⟵ KEY

↳ converted into lin. alg.

Linear = Lin. comb of X, Y
Quadr = Lin comb of X^2, XY, Y^2

$$\left.\begin{array}{ll}(X^2 - sY^2) \cdot X & \text{or} \\ (X^2 - sY^2) \cdot Y & \text{or} \\ (sX + tY) \cdot X^2 & \text{or} \\ (sX + tY) \cdot XY & \text{or} \\ (sX + tY) \cdot Y^2 & \end{array}\right\} \text{or any lin. comb.}$$

When do these 5 forms
↳ which values of (s,t)
generate all cubics?

$$
\begin{array}{c|ccccc|}
x^3 & 1 & & s & & \\
x^2y & & 1 & t & s & \\
xy^2 & -s & & & t & s \\
y^3 & & -s & & & t
\end{array}
$$

For which (s,t) do those 5 columns span the full 4 dim space?

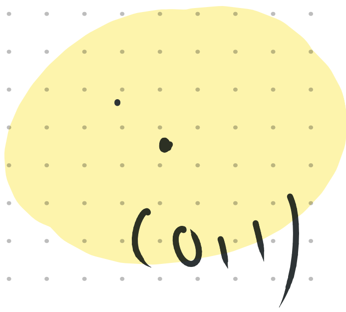
They do when $(s,t) = (0,1)$
$\iff \exists$ $\underline{4\times 4}$ non-deg block here.

$$\to \frac{\det(1\ 2\ 4\ 5)}{T}$$

↳ poly in s,t $P(s,t)$

Non-zero at $\underline{(s,t)=(0,1)}$

$P(s,t) \neq 0$.



THIS IS TYPICALLY NOT HOW $\mathrm{Im}(Z)$ or its compl. is computed.

$\underline{\{V, W\}} \subset Gr(2,4) \times Gr(2,4)$

$V \cap W \neq 0$

$\Updownarrow$

$\exists$ 1-dim $L \subset V \cap W$

$$Gr(2,4) \times Gr(1,4) \supset Z$$

$\{ (V, L) \mid V \supset L \}$ closed.

$$Gr(2,4) \times Gr(2,4) \times Gr(1,4)$$

$\left\{ (V, W, L) \;\middle|\; \begin{array}{l} V \supset L \ \& \\ W \supset L \end{array} \right\}$ ← $\underline{\text{closed}}$

$\Leftarrow$ $\underline{\text{closed}}$. $= \{(V,W,L) \mid V \supset L\} \cap \{(V,W,L) \mid W \supset L\}$

$$\{(V, \underline{\underline{W}}, L) \mid V \supset L\} = \pi^{-1}(Z)$$

$$\subset Gr(2,4) \times Gr(2,4) \times Gr(1,4)$$

$$\Big\downarrow \pi \quad \text{forget } W$$

$$\underbrace{\{(V, L) \mid V \supset L\}}_{Z} \subset Gr(2,4) \times Gr(1,4)$$

$\boxed{\underline{\text{completness}}} \Rightarrow$ Image is closed!

$$\{(V, W, L) \mid V \supset L \,\&\, W \supset L\}$$

$$\subset Gr(2,4) \times Gr(2,4) \times Gr(1,4)$$

$$\Big\downarrow$$

$$Gr(2,4) \times Gr(2,4)$$

Why are $Gr(2,n)$ Lines?

2 dim subs $\subset K^n$

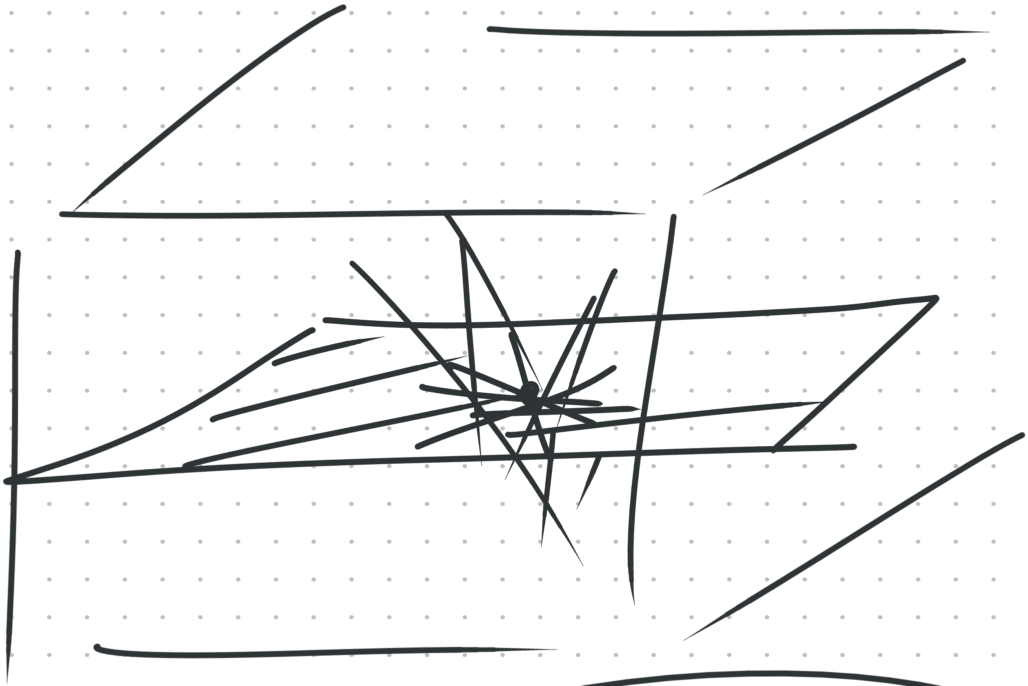
/scaling (throw away 0)

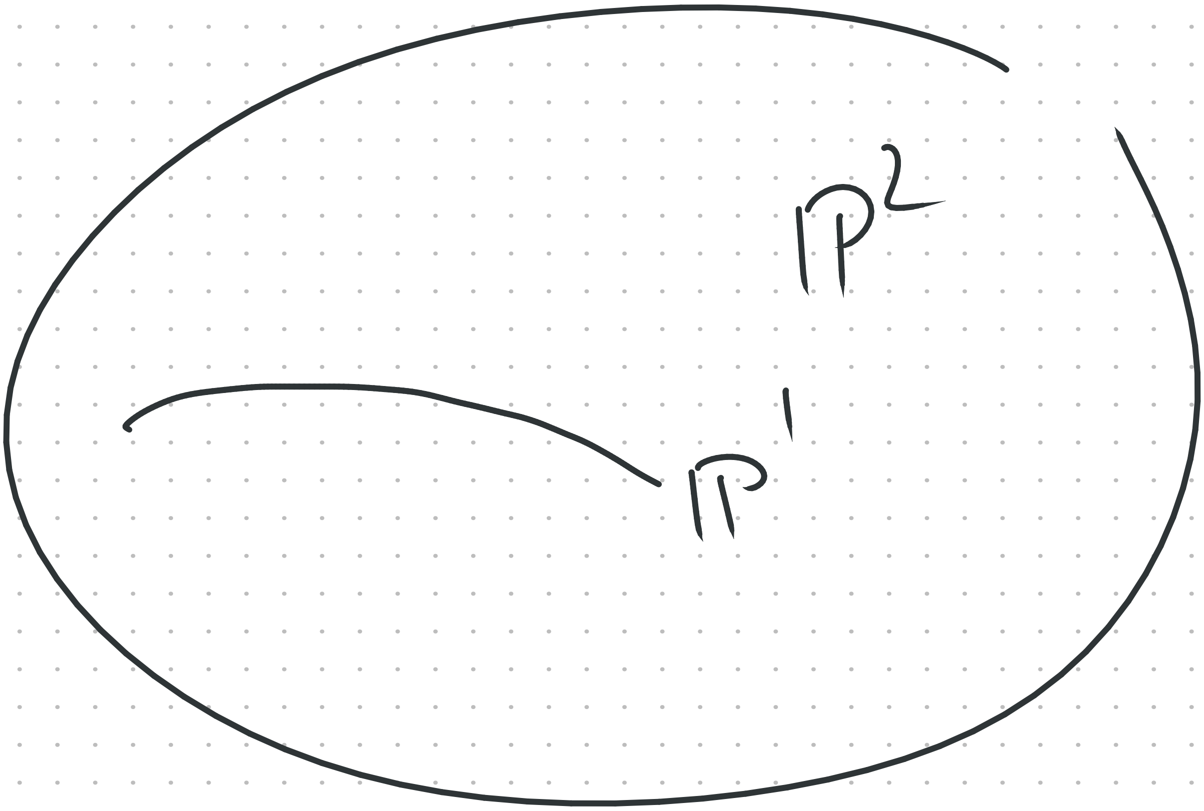
$\mathbb{P}^1 \subset \mathbb{P}^{n-1}$ "linear subspace"

$Gr(r,n)$

Vector spaces $K^r \subset K^n$

proj spaces $\mathbb{P}^{r-1} \subset \mathbb{P}^{n-1}$




$\mathbb{P}^2$
$\mathbb{P}^1$

- Spaces of projective lines in $\mathbb{P}^n$ $\quad$ " $\mathbb{P}^1$

$$\|$$

$$Gr(2, n+1)$$

- Spaces of projective planes $= \mathbb{P}^2$ in $\mathbb{P}^n$

$$\|$$

$$Gr(3, n+1)$$

- $Gr(1, n+1)$ "space of pts $= \mathbb{P}^0$ in $\mathbb{P}^n$"

$$\|$$

$$\mathbb{P}^n$$

Schemes :-

- A generalization of a variety.
- key new idea :-
  nilpotent functions are allowed.

- Want to have a space whose ring of functions is ~~k-valued~~

$$k[x]/x^2.$$

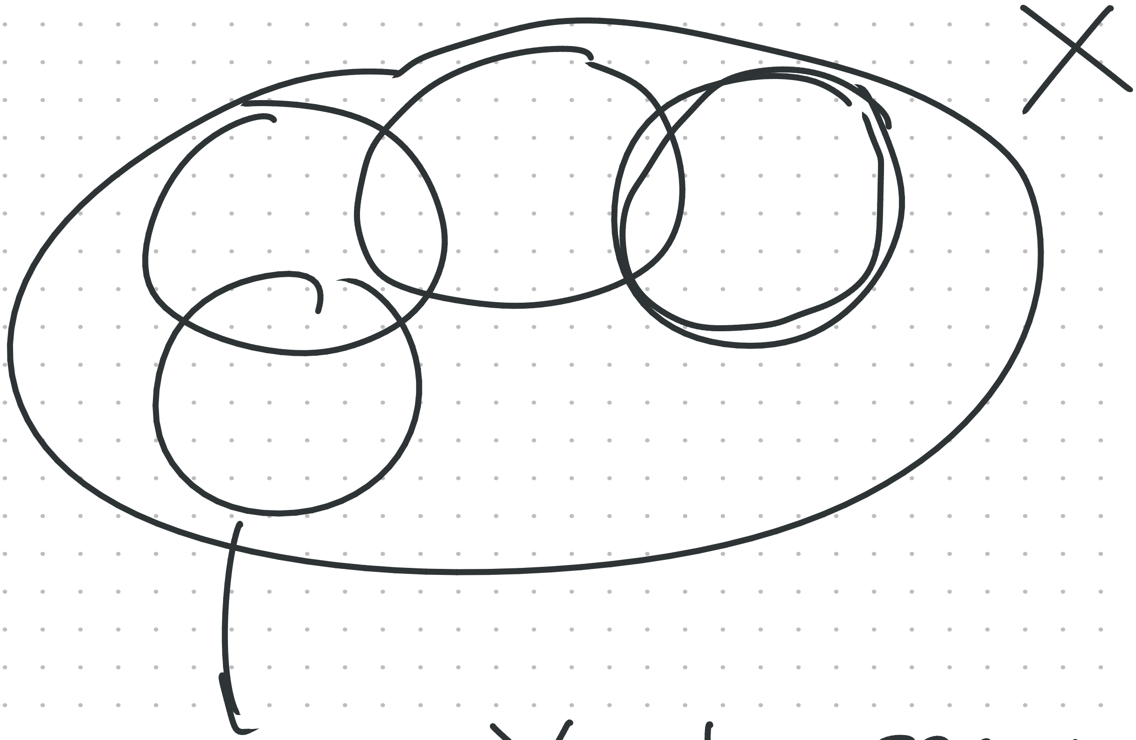
- Then space has two pieces of data

X top space

$+$

Data of "regular functions"

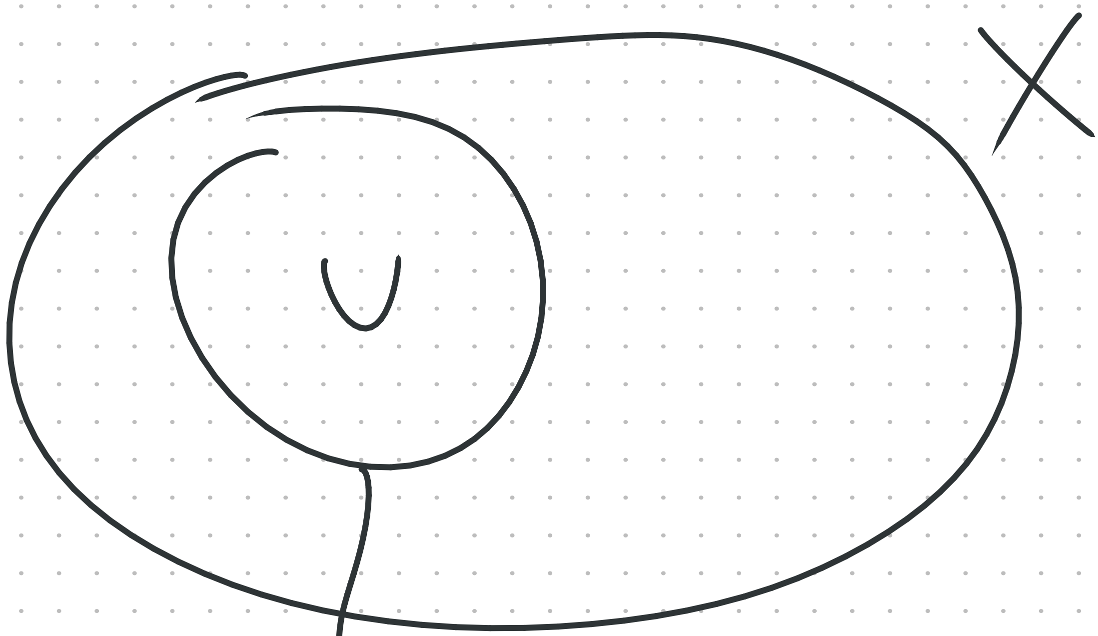
To be provided



Tell me X top space
+ $\forall$ open $U \subset X$ tell me
what ring $\mathcal{O}(U)$ is.

↳ there are conditions...

"Ringed space"



$\mathcal{O}(U) := k[U]$

gives a scheme.

<u>flexible</u>.

X

$\bullet := k$

$:= k[x]/x^2$

$:= k[x,y]/x^2, xy, y^2$

$:= \mathbb{Z}/p^2$

Spec(R) ← any ring!

Scheme

Varieties (we have defined)

↪ reduced schemes over an alg field k locally of finite type

Variety $\rightsquigarrow$ is a scheme

$A = k[x]$

$X \hookrightarrow \boxed{A \to k[x]/x^2}$

"$\operatorname{Spec} k[x]/x^2$" $\longrightarrow X$