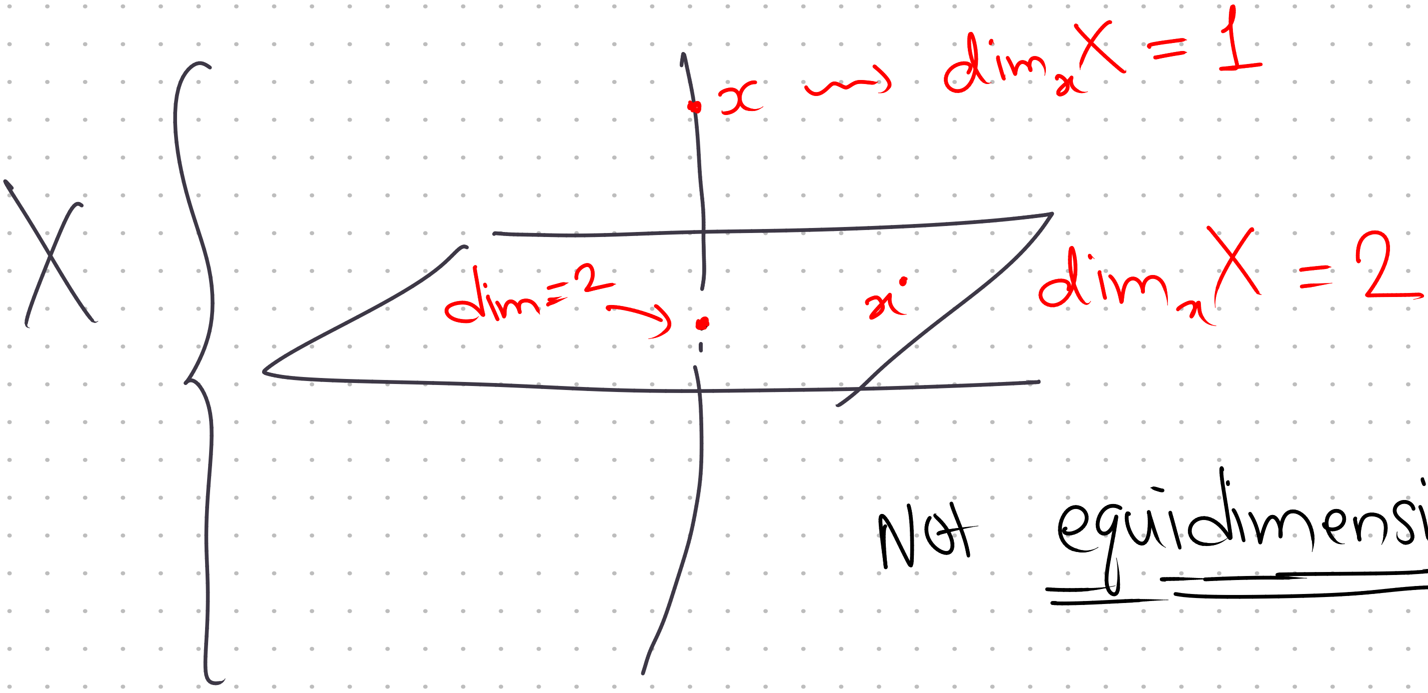
Dimension

Three definitions, all equivalent.

X a variety $x \in X$ point

$\dim_x X \leftarrow$ Dim of X at x

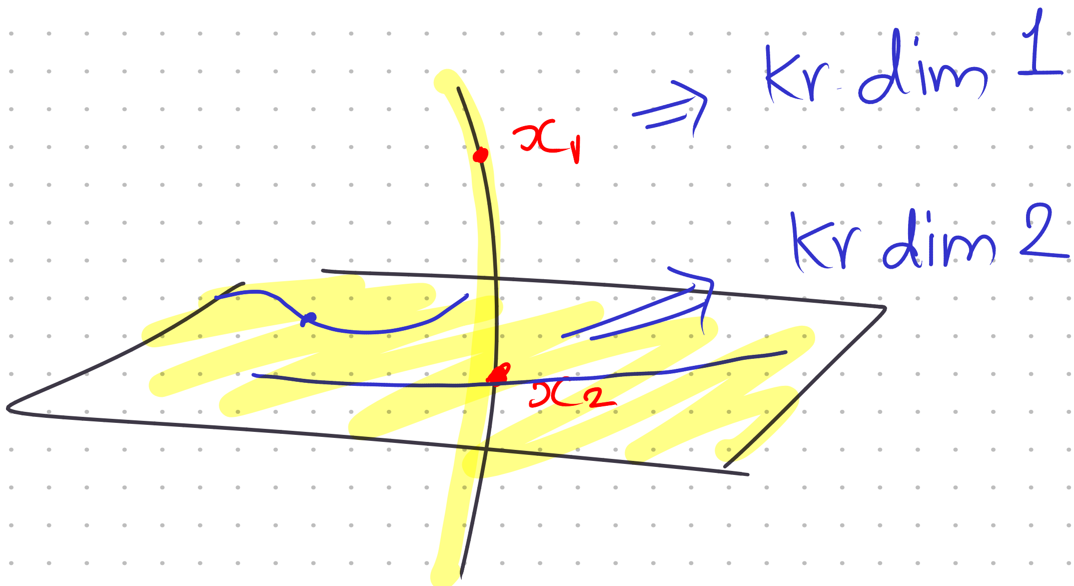


Not <u>equidimensional</u>

Def 1 : Krull dimension

$\mathrm{Krdim}_x X$ is the largest n s.t. there exists a chain of irred. closed subs of X starting at $\{x\}$ of length n.

$$\{x\} \subsetneq X_1 \subsetneq X_2 \cdots \subsetneq X_n$$



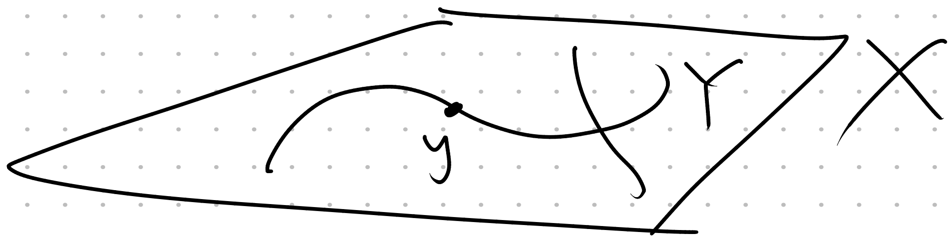
Rem: ① If X is irred.

then the longest chain will end with X

$\{x\} \subset \cdots\cdots \subset X_n = X$

② X irred $Y \subset X$ closed.

$y \in Y$



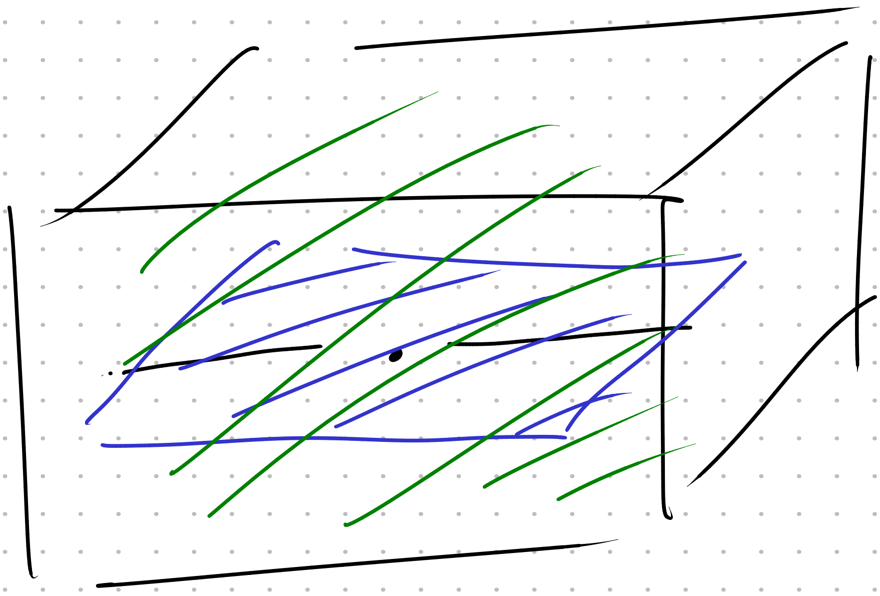
$\text{Krdim}_y Y < \text{Krdim}_y X$

chain ⤳ append X to get a longer one.

③ $\text{Kr-dim}_p \mathbb{A}^1 =$ equidim

$\{p\} \subset \mathbb{A}^1$ ⟹ Krdim 1

④ $\mathrm{Kr}\dim_0 \mathbb{A}^n \geq n$



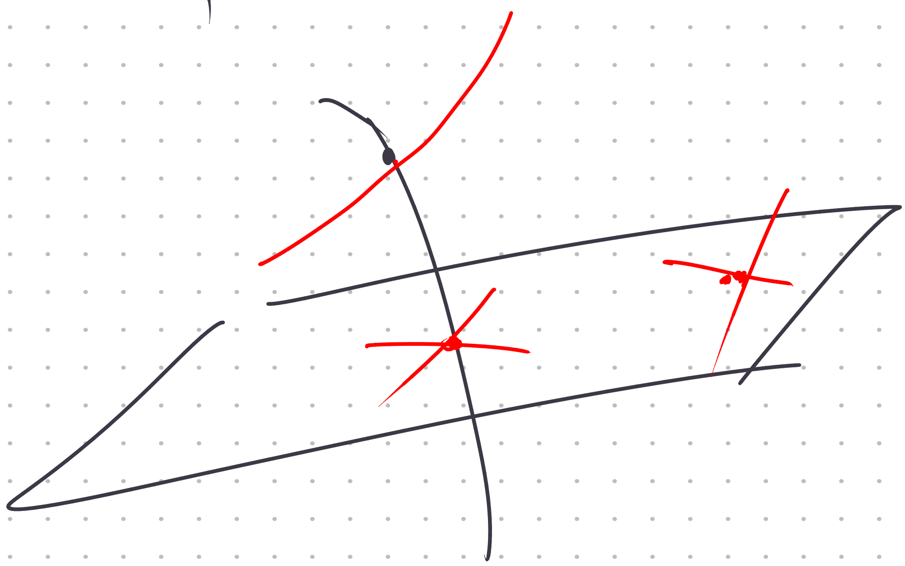
$$\| \quad 0 \subset \underbrace{\mathbb{A}^1 \subset \mathbb{A}^2 \subset \cdots \subset \mathbb{A}^n}_{\underline{\underline{n}}}$$

(Exercise - translate into algebra
X affine $A = k[X]$
$\leadsto$ chains of prime ideals)

Def 2: Slicing dimension

$x \in X$ has sl-dim n if n is the smallest s.t. $\exists$ open $U \subset X$ containing x and reg fun. $f_1, \dots, f_n$ such that the zero locus of $f_1, \dots, f_n$ on U is $\{x\}$.

"Need n functions to slice down the space to $\{x\}$".



Prop: X any and $Y = V(\underline{f})$ where $\underline{f} : X \to k$ reg fun. $y \in Y$

Then $\operatorname{Sldim}_y Y + 1 \geq \operatorname{Sldim}_y X$

Principal ideal theorem

Pf: Wlog X affine $X \subset \mathbb{A}^n$
so reg fun are poly.
$f_1, \dots, f_n$ that cut down Y to y.
$\underline{f}, \underline{f_1}, \dots, \underline{f_n}$, cut down X to y
$\Rightarrow$ Need at most 1 more func to chop down X.

(1) 

$\mathbb{A}^1$

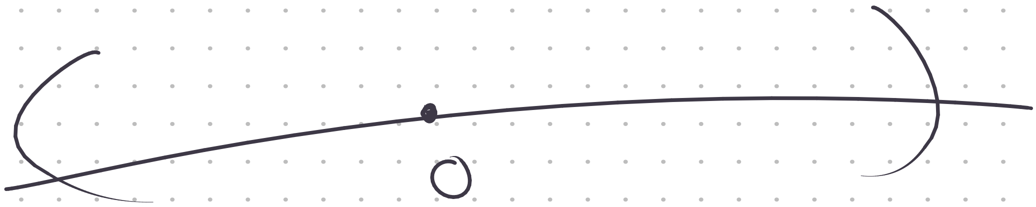
$f_1 = (x-p) \Rightarrow \mathrm{Sl.\,dim}_p \, \mathbb{A}^1 = 1$

(2) $\mathbb{A}^n$ $\quad x = (0, \dots, 0)$

$\mathrm{Sl\,dim}_x \, \mathbb{A}^n \le n$

$x_1, \dots, x_n \leftarrow$ zero locus is $(0, \dots, 0)$

(3) $\mathrm{Sl\,dim}_{[0:1]} \, \mathbb{P}^1 = \mathrm{Sl.\,dim}_0 \, \mathbb{A}^1$



Def 3 : Transcendental dim

Only applies to X irred

Does not dep. on $x \in X$

$k(X)$ is a field extn of k

$$\begin{array}{c} k(X) \\ \cup \\ k \end{array}$$

$\operatorname{Trdim} X :=$ Transcendence deg of $k(X)/k$.

"How many truly indep fun are there on X"

Transc. deg.

L/k a field ext.

Tr. deg (L/k) is largest n such that $\exists$ $l_1, \dots, l_n \in L$ which are algebraically indep over k.

↳ They do not satisfy any

$$p(l_1, \dots, l_n) = 0 \quad \text{where}$$

$$p \in k[x_1, \dots, x_n].$$

Ex. $\mathbb{C}/\mathbb{Q}$ ← inf. tr. deg

$\pi, \sqrt{\pi}$ are not alg. indep.

$x_1 \quad x_2$

alg. dep.

$$\boxed{x_2^2 - x_1 = 0}$$

Turns out:

π, e are alg. indep.

$\mathbb{C}(t)/\mathbb{C}$ tr. deg 1.

t

$\mathbb{C}(s,t)/\mathbb{C}$ tr. deg 2

s, t, ℓ

$k(x_1, \dots, x_n)/k$ tr. deg n.

$x_1, \dots, x_n$

$f : \underline{X} \dashrightarrow \underline{\underline{Y}}$ dominant rat map

$\hookrightarrow f(X) \subset Y$ dense

then

$$\underline{\underline{k(X)}} \supset \underline{\underline{k(Y)}} \supset k$$

$$\text{tr.deg}\,(k(Y)/k) \leq \text{tr.deg}\,(k(X)/k)$$

$$\Rightarrow \text{tr.dim}\,Y \leq \text{tr.dim}\,X$$

("No space filling curves in alg. geometry")

1. Kr dim x, X
2. Sl dim x, X
3. tr dim X (irred)

<u>Thm</u>: All these are equal. || Not proving this

For any X & $x \in X$

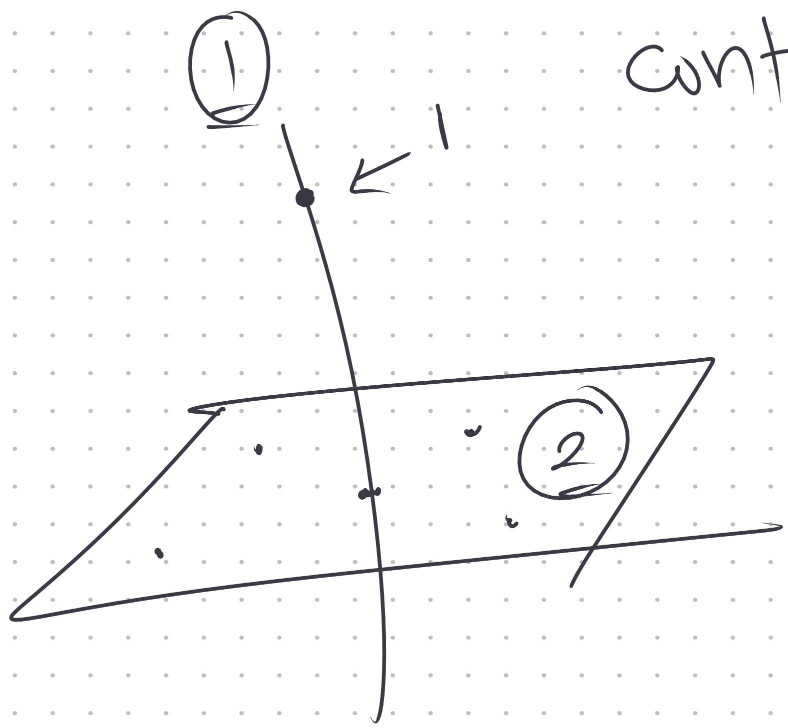
$$\mathrm{Kr\,dim}_x X = \mathrm{Sl\,dim}_x X = \underline{\mathrm{tr\,dim}}\, X \quad \leftarrow \cancel{x}$$

if $\underline{X \text{ irred}}$

<u>Cor</u> :- X irred then it is <u>equidim</u>. $\underline{\underline{\dim X}}$

In general:

$$\dim_x X = \max \text{ of } \dim \text{ of irred comp of } X \text{ containing } x$$



X

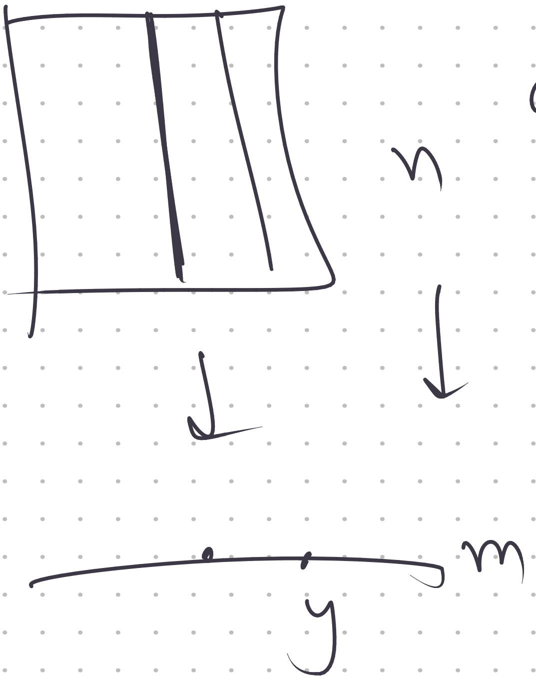
Thm (Dim of fibers)

$f: X \to Y$ $\quad$ X, Y irred

f dominant

X: dim n $\quad$ Y: dim m

Then



$$\dim f^{-1}(y) = n - m \quad *$$

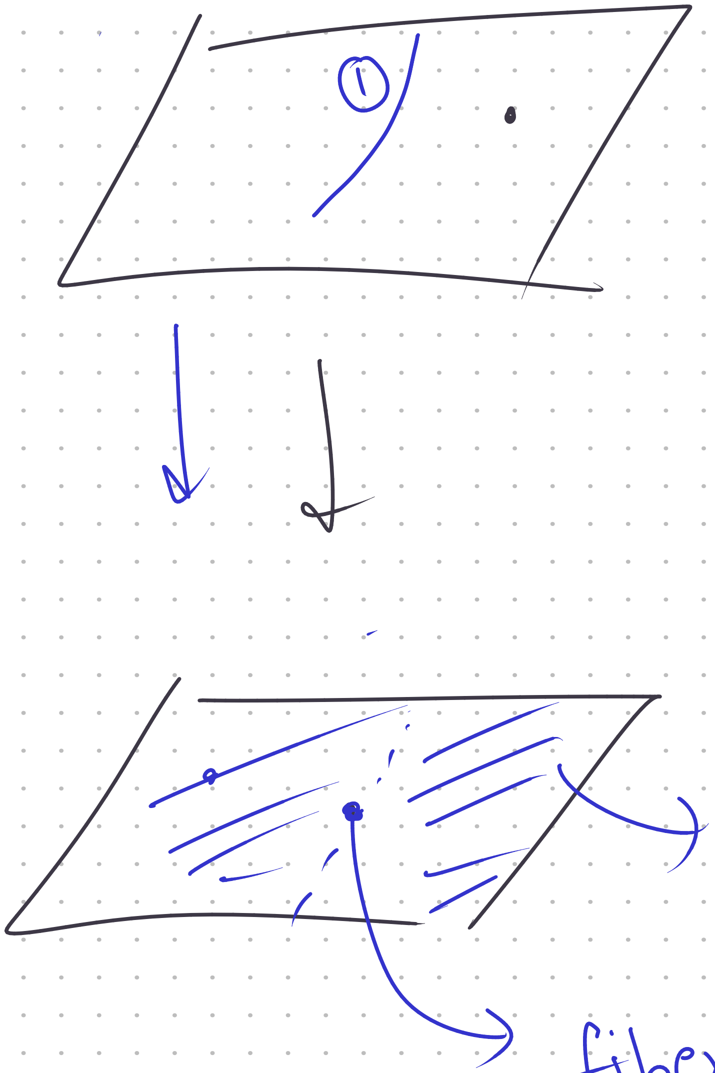
$*$ for all y in a Zariski open in Y

$$\dim f^{-1}(y) \geq n - m$$

for all y.

$$\mathbb{A}^2 \to \mathbb{A}^2$$

$$(x,y) \longmapsto (x, xy)$$



$$(3, 2/3) \to (3,2)$$

$$\overline{\left(a, \tfrac{b}{a}\right)} \to (a,b)$$

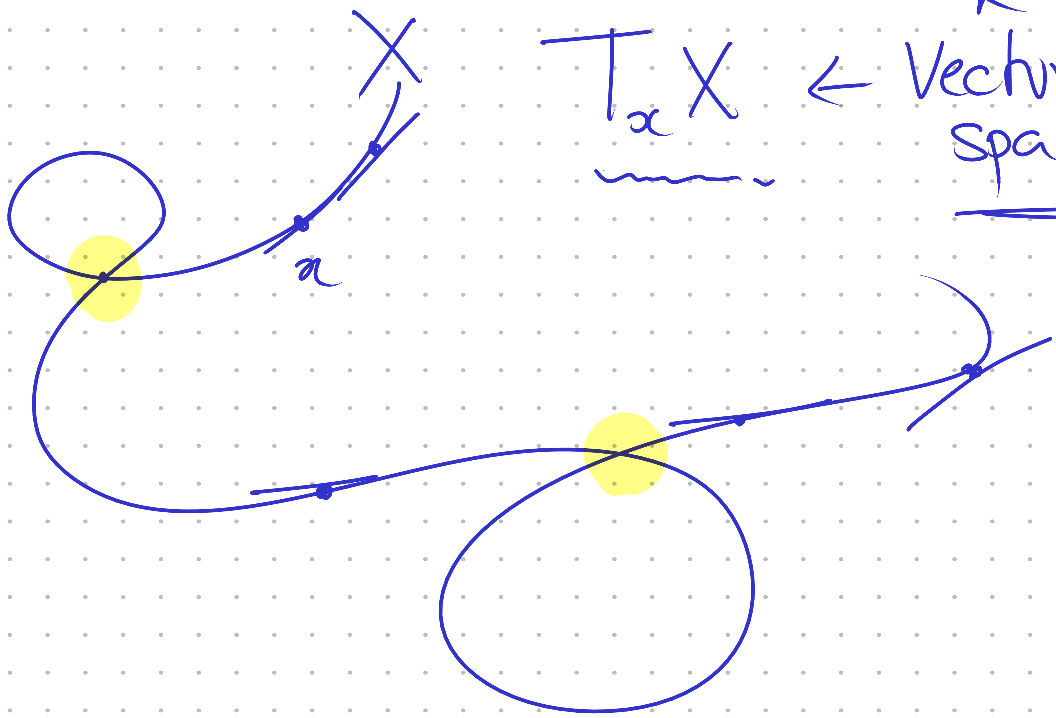
$$a \neq 0$$

$$(0, *) \to (0,0)$$

<u>exp dim fibers.</u>

fiber dim higher than expected.

$T_x X \leftarrow k\text{-Vector space}$



$$\dim(T_x X) \geq \dim_x X$$