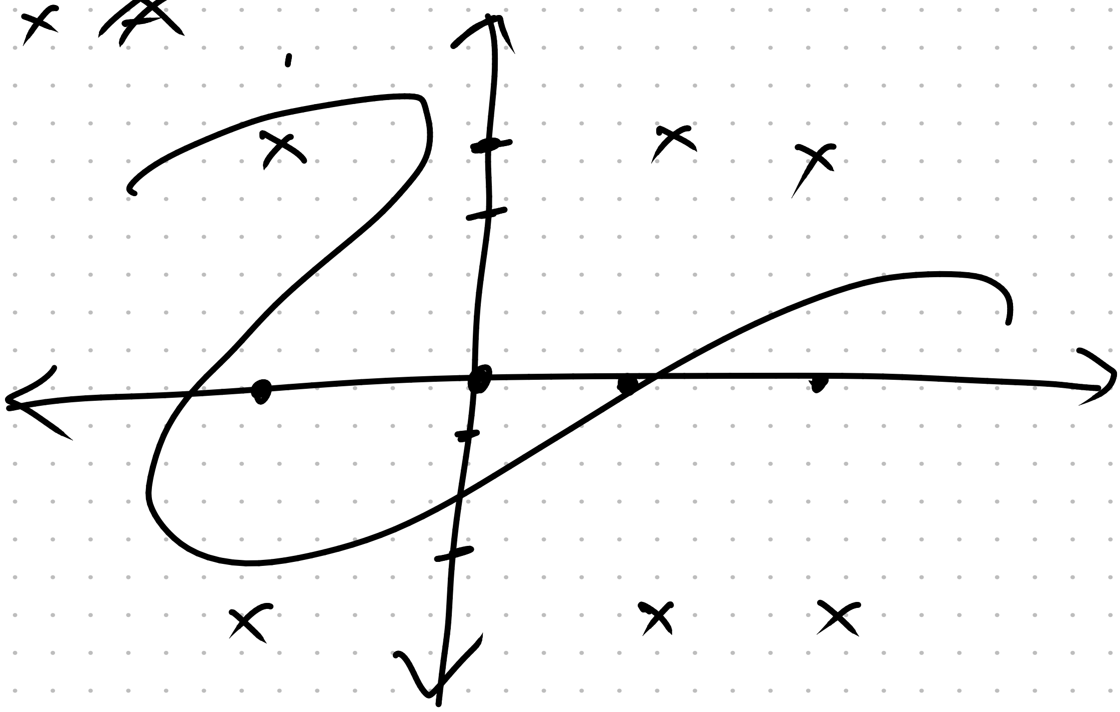
Products

X and Y alg. varieties Then
$X \times Y$ also an algebraic variety.

$\mathbb{A}^n \times \mathbb{A}^m \cong \mathbb{A}^{n+m}$ ↳ alg variety.

Caution: Zariski top on $\mathbb{A}^{n+m}$
is not the product top.

$\mathbb{A}^1 \times \mathbb{A}^1$



<u>Verify</u> $X \subset \mathbb{A}^n$ closed / open
$Y \subset \mathbb{A}^m$ closed / open

$\Rightarrow X \times Y \subset \mathbb{A}^{n+m}$ is closed / open

$X \subset \mathbb{A}^n$ locally closed
$Y \subset \mathbb{A}^m$ locally closed

$\Rightarrow X \times Y \subset \mathbb{A}^{n+m}$ is locally closed

$\hookrightarrow$ and hence a quasi-affine var

In general,

X has an atlas $\phi_i : U_i \to \underline{V_i}$

Y —"— $\phi_i' : U_i' \to \underline{\underline{V_i'}}$

$\underline{X \times Y}$ is covered by

$\underline{U_i \times U_j'} \longrightarrow \underline{V_i \times V_j'}$

Give it Zariski top coming from

Topologise so that $U_i \times U_j'$ is an open cover

$Z \subset X \times Y$ is $\underline{\text{closed}}$ if

$Z \cap (U_i \times U_j') \subset U_i \times U_j'$ is closed $\forall\ i,j$

Makes $X \times Y$ a top. space and give it the atlas

$$\phi_i \times \phi_j' : U_i \times U_j' \to V_i \times V_j'$$

$\rightsquigarrow$ $X \times Y$ becomes an alg var. with the product atlas

$$\begin{tikzcd}
Z \arrow[r] \arrow[rr, bend left] \arrow[dr, bend right] & X \times Y \arrow[r] \arrow[d] & Y \\
 & X &
\end{tikzcd}$$

both projections are regular.

Conversely

$$\begin{tikzcd}
Z \arrow[rr, bend left, "g"] \arrow[dr, "{(f,g)}"] \arrow[dd, bend right, "f"'] & & Y \\
& X \times Y & \\
X & &
\end{tikzcd}$$

f, g regular $\Leftrightarrow$ (f,g) is regular

This property characterises $X \times Y$.

Examples

$\mathbb{P}^n \times \mathbb{P}^m$ Charts $\leftarrow$ $(n+1)(m+1)$ copies of $\mathbb{A}^{n+m}$

$[X_0 : \cdots : X_n]$, $[Y_0 : \cdots : Y_m]$

Eg. $X_0 \neq 0$, $Y_0 \neq 0$

$[1 : x_1 : \cdots : x_n]$, $[1 : y_1 : \cdots : y_m]$

$$\updownarrow (x_1, \ldots, x_n) \qquad \updownarrow (y_1, \ldots, y_m)$$

$\mathbb{A}^n$ $\quad (x_1, \ldots, x_n, y_1, \ldots, y_m)$ $\quad \mathbb{A}^m$

$\mathbb{A}^{n+m}$

How do we construct closed subsets of $\mathbb{P}^n \times \mathbb{P}^m$?

$[X_0 : \cdots : X_n]$ $\quad$ $[Y_0 : \cdots : Y_m]$

$$F(\underbrace{X_0, \ldots, X_n}_{\lambda}, \underbrace{Y_0, \ldots, Y_m}_{\mu}) = 0$$

Bi-homog. polynomial. bi-deg (a,b)

Ex $\quad X_0^2 Y_1 + X_1 X_2 Y_3$

bi-hom poly of bidegree $(2,1)$

$$X_0 Y_1^3 + X_2 Y_2 Y_1^2$$

bihom of bideg $(1,3)$

$X_0^2 Y_1 + X_0 Y_1^2$ is not bi-homog.

$$F(\lambda X_0, \dots, \lambda X_n, \mu Y_0, \dots, \mu Y_m)$$
$$= \underline{\lambda}^a \underline{\mu}^b F(X_0, \dots, X_n, Y_0, \dots, Y_m)$$

F bihomog $\Rightarrow$ $V(F) \subset \mathbb{P}^n \times \mathbb{P}^m$ closed.

Def: A quasi-proj variety is a locally closed subset of $\mathbb{P}^n$

(or more generally, an alg var isomorphic to a locally closed subset of $\mathbb{P}^n$)

Locally closed

$= \text{Open} \cap \text{Closed}$

Open $\qquad\qquad$ Closed

Almost all varieties one sees are quasi-proj.

Products:

$\mathbb{P}^n \times \mathbb{P}^m \not\cong \mathbb{P}^{n+m}$

Turns out that $\mathbb{P}^n \times \mathbb{P}^m$ is isomorphic to a proj. variety.

As a result

$\underset{\text{q. proj}}{X} \times \underset{\text{q. proj}}{Y} \leftarrow$ q. Proj.

$X \subset \mathbb{P}^n$
$Y \subset \mathbb{P}^m$ $\Rightarrow$ $X \times Y \subset \mathbb{P}^n \times \mathbb{P}^m$

iso

closed of $\mathbb{P}^N$

Segre Embedding

$$\mathbb{P}^n \times \mathbb{P}^m \xrightarrow{\sigma} \mathbb{P}^N$$

$$N = (n+1)(m+1) - 1$$

$$[X_0 : \cdots : X_n], \; [Y_0 : \cdots : Y_m]$$

$$\longmapsto [X_0Y_0 : X_0Y_1 : \cdots \cdots : X_iY_j : \cdots : X_nY_m]$$

σ is regular ✓

Image is closed Z

$\sigma : \mathbb{P}^n \times \mathbb{P}^m \xrightarrow{\sim} Z$ iso.

Imagine $X = \begin{bmatrix} X_0 \\ \vdots \\ X_n \end{bmatrix}$

$Y = [Y_0 : \cdots : Y_m]$

$(n+1) \times (m+1)$ matrix.

$$\sigma(X, Y) = XY = \begin{bmatrix} & & \\ & X_i Y_j & \\ & & \end{bmatrix}$$

$\underline{XY}$ has rank 1.

Image of $\sigma \subset$ Rank 1 matrices/scaling.

Zero locus of all $\underline{2 \times 2}$ $\underline{\text{minors}}$

Z = Rank 1 $(n+1)\times(m+1)$ matrices / scaling.

$Z = V($ 2x2-minors$)$

$\subset \mathbb{P}^N \leftarrow \mathbb{P}$ (Matrices)

Any rank1 matrix can be expressed as XY where

X is a column vect.

Y is a row vect.

Unique up to scaling.

$\sigma : \mathbb{P}^n \times \mathbb{P}^m \longrightarrow Z$ bijection

$$\begin{tikzcd} & Z \\ (X, Y) & M \arrow[l, "\text{regular}"'] \end{tikzcd} \quad (\text{rank } 1)$$

Example

$$\mathbb{P}^1 \times \mathbb{P}^1 \xrightarrow{\sigma} \mathbb{P}^3$$

$$\mathbb{P}^3 = \mathbb{P}(2 \times 2 \text{ matrices})$$

$$\begin{pmatrix} X & Y \\ Z & W \end{pmatrix}$$

$$\text{Image} = V(XW - ZY)$$

V(Quadratic) in $\mathbb{P}^3$ is isomorphic to $\underline{\underline{\mathbb{P}^1 \times \mathbb{P}^1}}$

We showed $XW - ZY$

"Almostall" quadratics can be brought into this form by a linear change of <u>coordinates</u>.

$$V(Q) \subset \mathbb{P}^2 \cong \mathbb{P}^1$$

$$V(Q) \subset \mathbb{P}^3 \cong \mathbb{P}^1 \times \mathbb{P}^1$$

Story :-

We understand

V (Linear) ✓

V (Quadratic) ✓

V (cubic) ← mysterious.

$V(\text{cubic}) \subset \mathbb{P}^2$ Early 1800

$\subset \mathbb{P}^3$ Late 1800

$\subset \mathbb{P}^4$ 1974

$V(\underline{\text{cubic}}) \subset \mathbb{P}^5$ open.

$\vdots$

Claire Voisin (2015-ish)

$V(\mathrm{Bihom}(2,1))$

$\subset \mathbb{P}^2 \times \mathbb{P}^3$