Regular functions on $\mathbb{P}^1$

④ $k[\mathbb{P}^1] = k$

How to compute $k[X]$?

- X affine then ✓
- Reduce to the affine situation
  ↳ Patch & glue.

$$\mathbb{P}^1 = U_0 \cup U_1$$

$$U_0 = \{[1:y]\} \xrightarrow{y} \mathbb{A}^1, \quad U_1 = \{[x:1]\} \xrightarrow{x} \mathbb{A}^1$$

$$\mathbb{A}^1 \cup \mathbb{A}^1 = \mathbb{P}^1$$

$$U_0 \cap U_1 = \mathbb{P}^1 \setminus \{[0:1], [1:0]\} = \mathbb{A}^1 \setminus \text{origin} \cong V(xy-1) \subset \mathbb{A}^2$$

affine

Function on $U_0 \cup U_1$

$=$ Function on U_0 & Function on U_1 which agree on $U_0 \cap U_1$

$$
\begin{tikzcd}
 & k[U_0] \arrow[dr, "\text{rest}"] & \\
k[\mathbb{P}^1] \arrow[ur] \arrow[dr] & & k[U_0 \cap U_1] \\
 & k[U_1] \arrow[ur] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
k[y] \arrow[dr, "y \mapsto y"] & \\
 & \dfrac{k[x,y]}{(xy-1)} \cong k[x, x^{-1}] \\
k[x] \arrow[ur, "x \mapsto x"'] &
\end{tikzcd}
$$

$$\frac{k[x,y]}{(xy-1)} \hookrightarrow k(x)$$

$$x \longmapsto x$$
$$y \longmapsto \frac{1}{x} = x^{-1}$$

Image $= k[x, x^{-1}]$!

$$\begin{tikzcd}
p(y) \in k[y] \arrow[dr] & \\
& k[x,x^{-1}] = k[U_0 \cap U_1] \\
q(x) \in k[x] \arrow[ur] &
\end{tikzcd}$$

$y \longmapsto x^{-1}$

$x \longmapsto x$

When is $p(x^{-1}) = q(x)$ as a function on $\mathbb{A}^1 \setminus \{0\}$

$p(x^{-1}) =$

$\underline{p_0} + p_1 x^{-1} + \dots + p_n x^{-n}$

$q(x) = \underline{q_0} + q_1 x + \dots + q_m x^m$

$p_0 + p_1 x^{-1} + \dots + p_n x^{-n}$
$= q_0 + q_1 x \dots + q_m x^m$

$\forall x \in \mathbb{A}^1 - \{0\}$ $\underline{k}$ $\underline{\text{alg}}$ $\underline{\text{closed}}$

Can only happen if corresp coeff are equal (minor argument)

$p_1 = p_2 = \dots = 0 \qquad p_0$

$q_1 = q_2 = \dots = 0 \qquad \overset{\shortparallel}{q_0}$

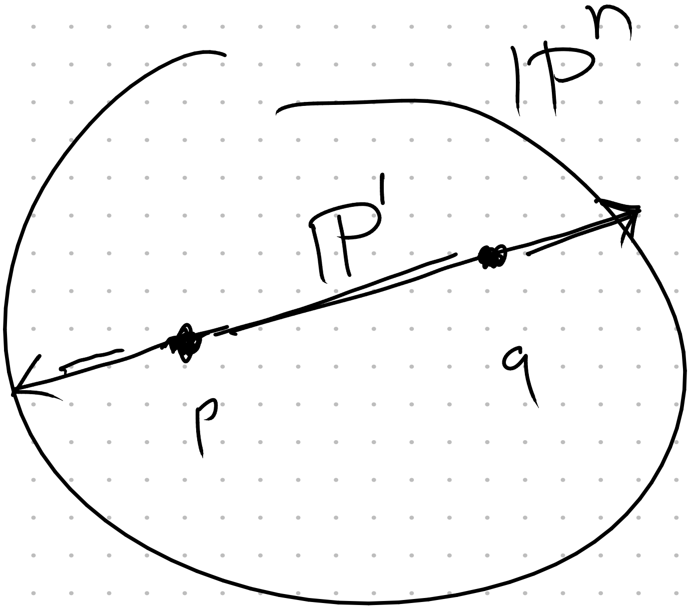
For $\mathbb{P}^n$

① Honest work.

(n+1) affines & their overlaps.

② Trick.

$\mathbb{P}^n \ni \underline{p}, \underline{q}$ distinct.



Through any two points passes a unique line.

↑

Through any two lin. ind vectors passes a unique 2 dim space

2dim $V \subset k^{n+1}$

Linear iso
$V \xrightarrow{\simeq} k^2$

Lines in yellow

$\underline{\mathbb{P}V}$ = Lines in V

$\|2$

$\mathbb{P}^1$

<u>Veronese</u> : $\varphi : \mathbb{P}^1 \hookrightarrow \mathbb{P}^n = [U_0 : \cdots : U_n]$

$$[X : Y] \mapsto [\underbrace{X^n : X^{n-1}Y : X^{n-2}Y^2 : \cdots : Y^n}_{\text{cannot be all zero.}}]$$

<u>Equations</u> : $U_i U_j - U_k U_l$ if $i+j = k+l$. (How many?)

I = ideal generated.

$\varphi : \mathbb{P}^1 \to V(I)$ $\quad U_i = X^i Y^{n-i}$

↳ makes all elts in $I \mapsto 0$.

Inverse (Regular)

$[U_0 : \cdots : U_n] \overset{\varphi}{\longmapsto} [U_0 : U_1]$,

on $V(I)$ or $[U_1 : U_2]$

or $[U_2 : U_3]$

$U_0 U_2 - U_1^2 = 0$ $\vdots$

& likewise for any of these.

Needs to be checked

① $\forall$ $p \in V(I)$ at least one is

$p = [U_0 : \cdots : U_n]$

at least one $U_i \neq 0$.

so $[U_{i-1} : U_i]$ or

$[U_i : U_{i+1}]$

well-defined, so p is in the domain of φ.

$\psi : V(I) \to \mathbb{P}^1$

(2)

$$\begin{tikzcd} \mathbb{P}^1 \arrow[r, "\varphi"] \arrow[rr, bend right, "id"'] & V(I) \arrow[r, "\psi"] & \mathbb{P}^1 \end{tikzcd}$$

$$[x:y] \mapsto [x^n : \dots : y^n] \mapsto [x:y]$$

(3) $V(I) \to \mathbb{P}^1 \to V(I)$

$$[U_0 : \dots : U_n] \mapsto [U_i : U_{i+1}]$$

$$\mapsto [U_i^n : U_i^{n-1} U_{i+1} : \dots\dots]$$

How does one check

$$[p_0 : \dots : p_n] = [q_0 : \dots : q_n]$$

iff $p_i q_j - p_j q_i = 0$

1

$$[U_0 : U_1 \cdots : U_n] \quad \cancel{\parallel} \quad [U_i^n : U_i^{n-1} U_{i+1} : \cdots\cdots] \qquad \parallel \quad ★$$

$$U_0 U_i^{n-1} U_{i+1} = U_1 U_i^n \quad ?$$

$$U_0 U_{i+1} = U_1 U_i \quad \text{YES}?!$$

Equations in the ideal force ★

1. $(p_0, \ldots, p_n), (q_0, \ldots, q_n)$ lin dep? $\Longleftrightarrow$

$$\operatorname{rk}\left(\begin{bmatrix} p_0 & \cdots & p_n \\ q_0 & \cdots & q_n \end{bmatrix}\right) = 1$$

$\Leftrightarrow$ All 2×2 minors vanish

M has rank $\le r$

$\iff$ All $(r+1)\times(r+1)$ minors vanish.

polynomial equations!

Same thm is true in higher dim.

eg $\mathbb{P}^n$ all mon. of deg m

$N = \binom{n+m}{n}$

Get $\mathbb{P}^n \xrightarrow{\varphi} \mathbb{P}^{N-1}$

$[x_0 : \cdots : x_n] \mapsto [\_ : \_ : \_]$

all deg m mons.

Im is closed

(can write explicit eq^n's)

φ is an iso on the image

$$\begin{array}{ccc} \mathbb{P}^1 & \longrightarrow & \mathbb{P}^2 \\ & & \cup \\ & & V(\underline{\underline{XY-Z^2}}) \end{array}$$

↳ Plane conic

Plane conic $\cong \mathbb{P}^1$