Projective Space

Want to verify transition maps are regular.

$\phi_1 \circ \phi_2^{-1}$

$\mathbb{P}^n = \{ [X_0 : \cdots : X_n] \mid \text{Not all zero} \} \,/\, \text{scaling}.$

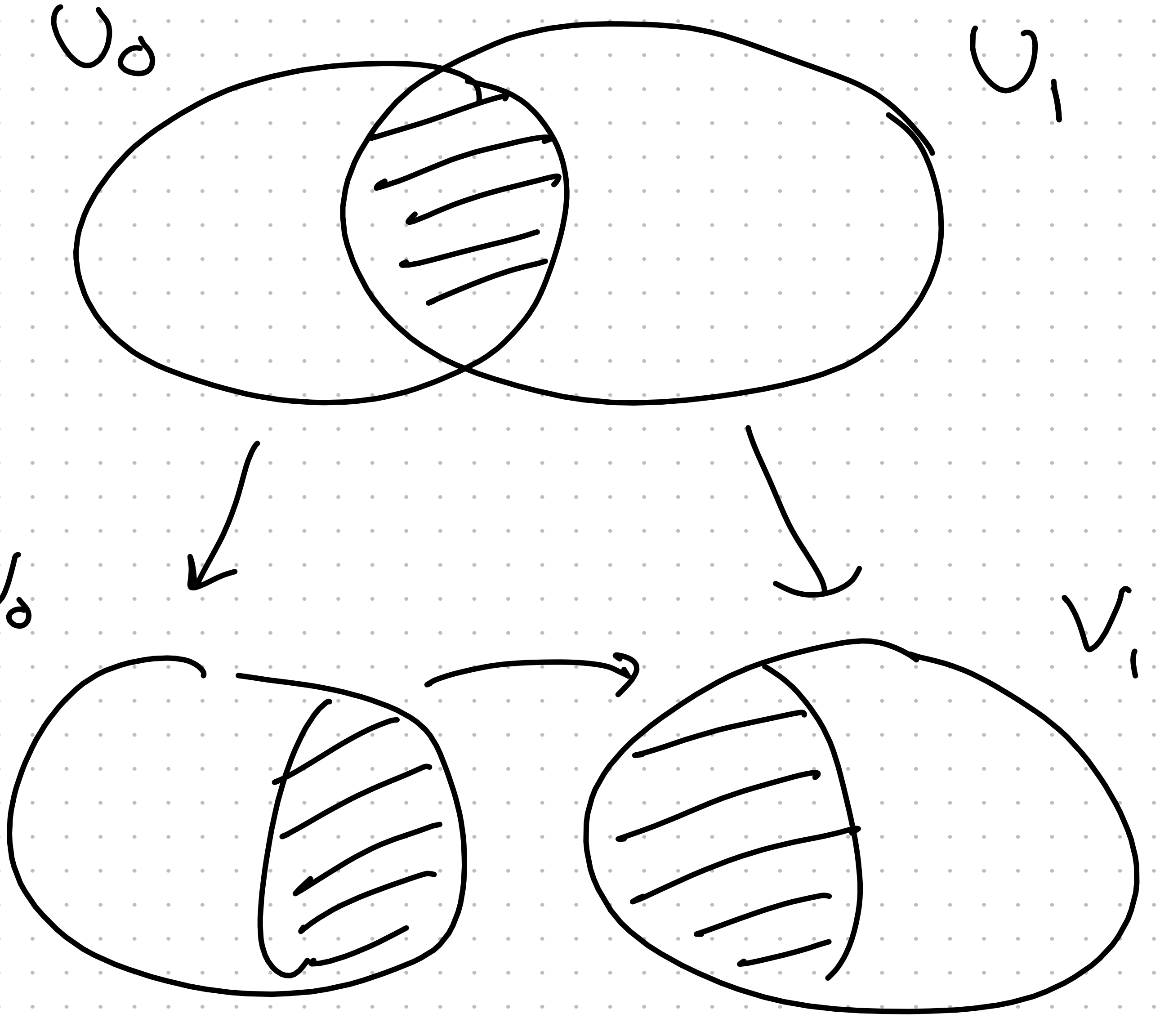
$U_i = \{ X_i \neq 0 \}$

$\phi_i : U_i \to \mathbb{A}^n$ compatible.

$$\begin{array}{ccc} U_0 = & \{ [x_0 : x_1 : \cdots x_n] \mid x_0 \neq 0 \} \\ \downarrow \phi_0 & \mapsto \phi_0 \\ \mathbb{A}^n \ni & \left(\frac{x_1}{x_0}, \cdots, \frac{x_n}{x_0} \right) \end{array}$$

$$U_1 = \{ [x_0 : x_1 : \cdots : x_n] \mid x_1 \neq 0 \}$$

$$\begin{tikzcd} U_1 \arrow[d, "\phi_1"] & {[x_0 : x_1 : \cdots : x_n]} \arrow[d, mapsto] \\ \mathbb{A}^n & \left(\frac{x_0}{x_1}, \ldots, \frac{x_n}{x_1} \right) \end{tikzcd}$$



$$\underline{\underline{U_0}} \cap U_1 = \left\{ [\underline{x_0} : \underline{x_1} : \cdots : x_n] \;\; \begin{matrix} x_0 \neq 0 \\ x_1 \neq 0 \end{matrix} \right\}$$

$$\begin{tikzcd}[column sep=small]
 & U_0 \cap U_1 \arrow[dl, shift left, "{\left(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right) = (a_1, \ldots, a_n)}"'] \arrow[dr, "{\left(\frac{x_0}{x_1}, \ldots, \frac{x_n}{x_1}\right) = (b_1, \ldots, b_n)}"] & \\
\mathbb{A}^n \supset \underline{\underline{\{a_1 \neq 0\}}} \arrow[ur, shift left] \arrow[rr] & & \underline{\underline{\{b_1 \neq 0\}}} \subset \mathbb{A}^n
\end{tikzcd}$$

$$\psi$$

$$\begin{tikzcd}[column sep=small]
 & {[1 : a_1 : \cdots : a_n]} \arrow[dr, mapsto] & \\
(a_1, \ldots, a_n) \arrow[ur, mapsto] \arrow[rr] & & \left(\frac{1}{a_1}, \frac{a_2}{a_1}, \ldots, \frac{a_n}{a_1}\right)
\end{tikzcd}$$

Regular? ✓

$$\left(\frac{1}{b_1}, \frac{b_2}{b_1}, \ldots, \frac{b_n}{b_1}\right) \longleftarrow (b_1, \ldots, b_n)$$

$\underline{\mathbb{P}^1} := \mathbb{A}^1 \cup \mathbb{A}^1$

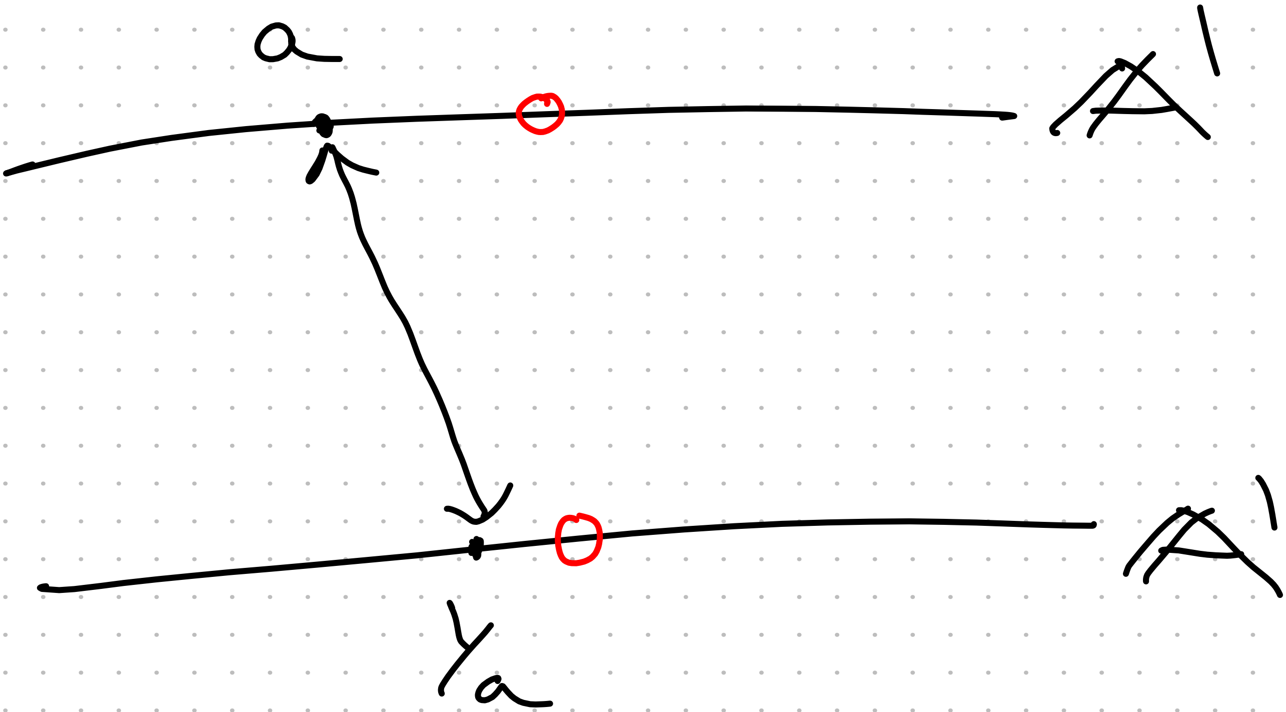
$[X:Y]$ $\quad \{X \neq 0\} \quad [1 : \boxed{\tfrac{Y}{X}}]$

$\quad \{Y \neq 0\} \quad [\boxed{\tfrac{X}{Y}} : 1]$

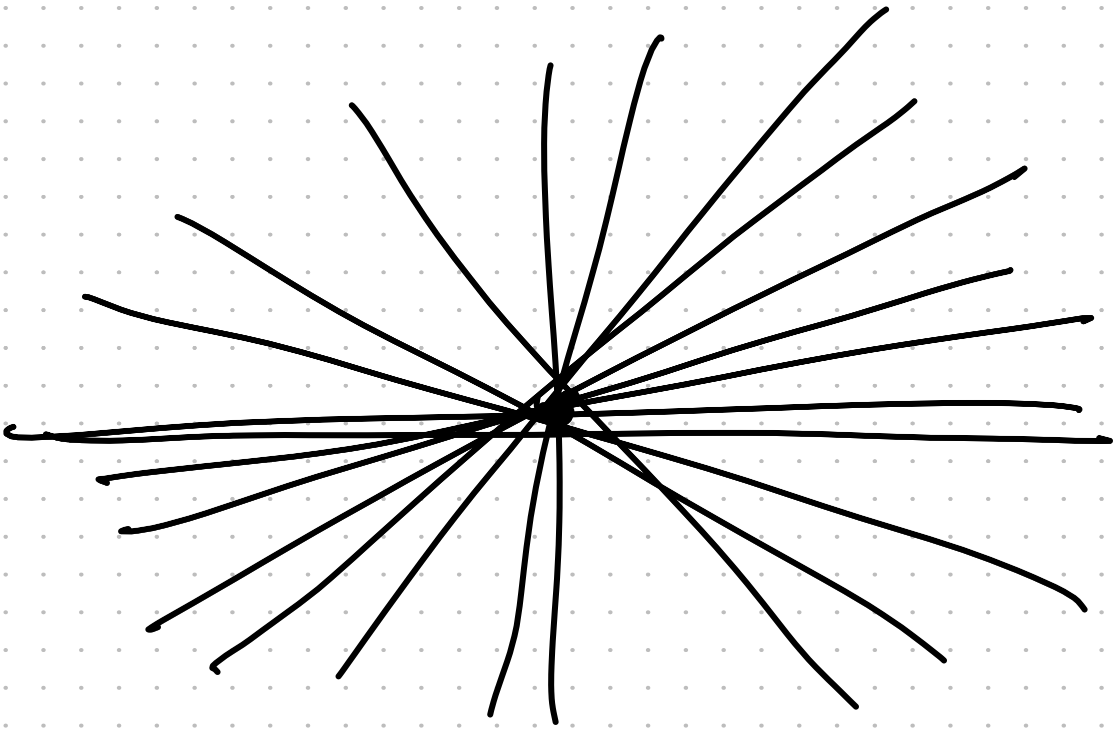
$$\begin{tikzcd} & U_0 \cap U_1 \arrow[dl, "Y/X"'] \arrow[dr, "X/Y"] & \\ \mathbb{A}^1 \arrow[rr] & & \mathbb{A}^1 \end{tikzcd}$$

$$a \longmapsto \frac{1}{a}$$

$\mathbb{P}^1$ = Obtained by gluing two $\mathbb{A}^1$s along $\mathbb{A}^1 \setminus \{0\}$ but by the inverse map



$\underline{\underline{\mathbb{P}^1}}.$ $\mathbb{P}^1$



(2) X alg. variety with atlas

$\phi_i : U_i \to V_i$

and $Y \subset X$ is closed

then Y is naturally an alg. variety. & atlas is

$\phi_i : U_i \cap Y \to \phi_i(U_i \cap Y)$

$\bigcup U_i = X \Rightarrow \bigcup (\underline{\underline{U_i \cap Y}}) = Y$

$U_i \subset X$ open

$U_i \cap Y \subset Y$ open

$$\begin{tikzcd}
\overline{\phi_i(U_i \cap Y)} \arrow[r, phantom, "\overset{\text{closed}}{\subset}"] & V_i \quad \text{Q.affine} \\
U_i \cap Y \arrow[u] \arrow[r, phantom, "\underset{\text{closed}}{\subset}"] & U_i \arrow[u, "\phi_i"']
\end{tikzcd}$$

Transition maps are restrictions of the original transition maps.

③ F homog. $\in k[x_0, \dots, x_n]$

Define $\mathbf{V}(F) \subset \mathbb{P}^n$

$\mathbf{V}(F) = \{ [x_0 : \dots : x_n] \mid F(x_0, \dots, x_n) = 0 \}$

F is NOT A FUNCTION

$\mathbb{P}^n \longrightarrow k$

$[x_0 : \dots : x_n] \sim [\lambda x_0 : \dots : \lambda x_n]$

$\lambda^d F(x_0, \dots, x_n) = F(\lambda x_0, \dots, \lambda x_n)$

$d = \deg F.$

Closed?

Two ways

① Quotient top

$$\mathbb{A}^{n+1} \setminus 0 \xrightarrow{\pi} \underline{\mathbb{P}^n}$$

$Z \subset \mathbb{P}^n$ is closed iff

$\pi^{-1}(Z) \subset \mathbb{A}^{n+1} \setminus 0$ is closed.

$\pi^{-1}(V(F))$ function on?

$= \underline{V(F) \text{ in } \mathbb{A}^{n+1} \setminus 0}$

② Charts

$X = \bigcup U_i$ open cover

Then $Z \subset X$ is closed

iff $\forall i : \underline{Z \cap U_i \subset U_i}$ is closed

$V(\underline{\underline{F}}) \subset \mathbb{P}^n$ restrict to $x_i \neq 0$.

$[x_0 : \cdots : x_n]$

$\cong$

$\left[\frac{x_0}{x_i} : \cdots : \frac{x_{i-1}}{x_i} : 1 : \frac{x_{i+1}}{x_i} : \cdots : x_n\right]$

$\updownarrow \quad a_j = \frac{x_j}{x_i}$

$\mathbb{A}^n \; \{ (a_0, a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n)$

$\updownarrow$

$V(F) \cap \mathbb{A}^n$ $\quad \boxed{[a_0 : \cdots : 1 : \cdots a_n]}$

$\shortparallel$

$\{ \underbrace{(a_0, \dots, a_n)}_{} \mid F(a_0, a_1, \dots, a_{i-1}, 1, a_{i+1}, \dots, a_n) = 0 \}$

④ $V(I)$ is very similar.

Take $V(I) \subset \mathbb{A}^{n+1} \setminus 0$ & set

$V(I) \subset \mathbb{P}^n$ to be the

image of $V(I) \subset \mathbb{A}^{n+1} \setminus 0$.

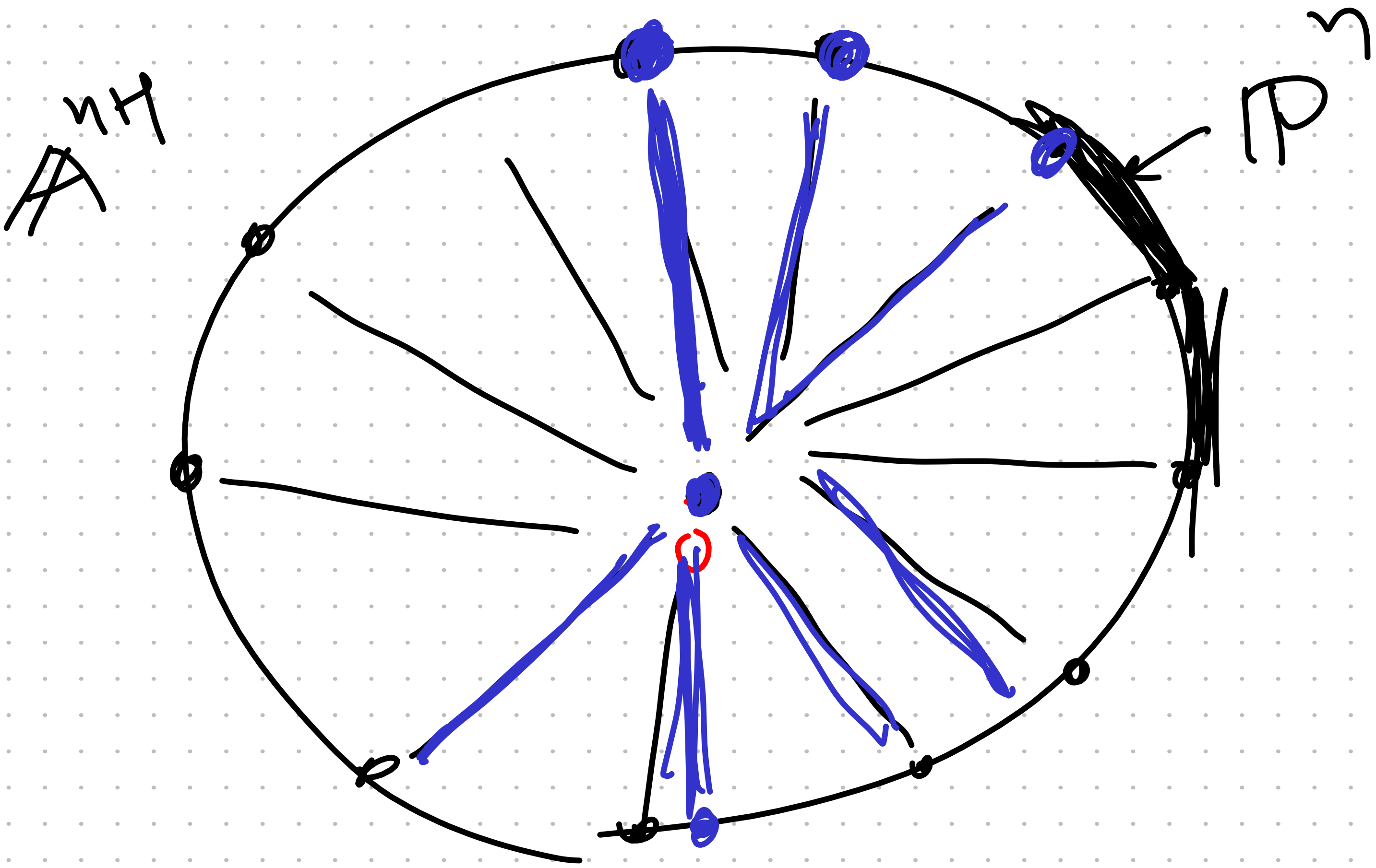
$$V(I) = \{ [X_0 : \cdots : X_n] \mid \Rightarrow F(X_0, \ldots, X_n) = 0 \ \forall \text{ homog. } F \in I \}$$

$$= \bigcap_{\text{homog } F \in I} V(F)$$

(5) Every closed $Z \subset \mathbb{P}^n$ is $V(I)$ for homog I.

$$\begin{array}{ccc} \mathbb{A}^{n+1}\setminus 0 & \xrightarrow{\pi} & \mathbb{P}^n \\ \cup \ \text{closed.} & & \cup \\ \pi^{-1}(Z) & \longrightarrow & Z \end{array}$$

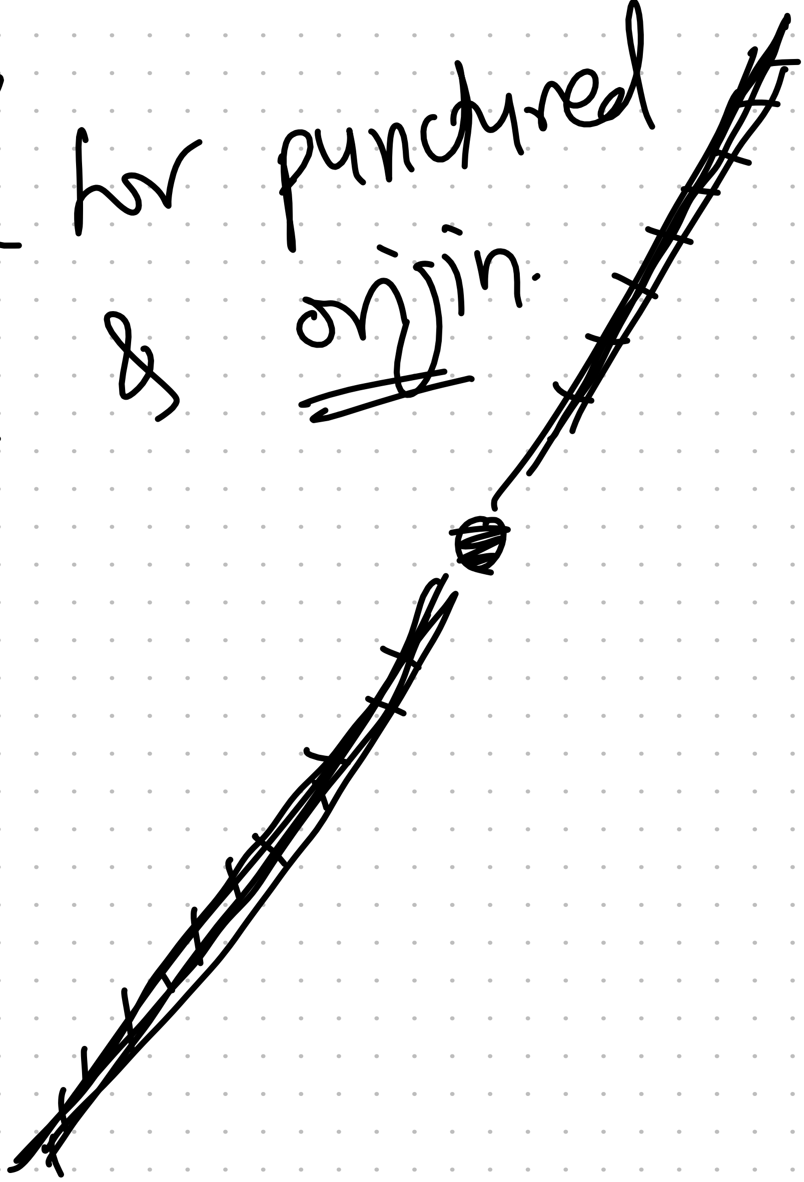
$Z' = \overline{\pi^{-1}(Z)}$ in $\mathbb{A}^{n+1}$.



Closure of S contains y

$\iff$ Any poly. vanishing on S also must vanish on y.

True for punctured line & origin.



$$Z' = \pi^{-1}(Z) \cup \{0\}$$

A cone (closed under scaling)

$$\| \quad V(I)$$

homog.

Follows that

$$Z = V(I) \subset \mathbb{P}^n.$$

Can happen:

$V(I) = \phi$ even if

$I \neq (1)$.

e.g. $I = \langle x_0, \ldots, x_n \rangle$

$V(I) \subset \mathbb{P}^n$ is ϕ.

(Exc. $V(I) = \phi$ iff

$\sqrt{I} = \left((x_0, \ldots, x_n) \right)$

or $\sqrt{I} = (1)$. $\|$